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MULTISPECIES GOMPERTZ-TYPE MODEL WITH TIME DEPENDENT PARAMETERS AND ITS STOCHASTIC GENERALIZATION

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A deterministic model for several-species-population which follows Gompertz' law, has been proposed and extended for the case of time-dependent periodic and non-periodic growth and interaction rates. The method of solutions has been given together with the explicit solutions for the two-species population. The stochastic generalization of this model has been proposed. The Fokker-Planck equation corresponding to the stochastic differential equations arising out of this model, has been approximately evaluated for the transition probabilities for both the stationary and non stationary cases.

1. INTRODUCTION

The deterministic models of population growth and of the interaction among the biological species are based on the phenomenological equations such as Verhulst Logistic equation, Lotka-Volterra equations and Gompertz' Law[6,9]. Again, in the stochastic model the environmental fluctuations on the macroscopic behaviour of the linear and non-linear system have been considered. Several mathematical tools are employed there; for example, the Markov process techniques and the stochastic differential equations, together with their conversions to the Fokker-Planck equations. In some cases, the interaction among the species has been taken as the interaction of Markov processes, whereas, in some other cases, the 'stochasticity' has been introduced by inserting the random parts in the parameters of the deterministic models and thus the stochastic differential equations are generated. In fact, in an earlier paper[3], the stochastic version of non-linear model of several interacting species in a randomly varying environment together with predator-prey type random interaction, has been investigated. There the Fokker-Planck equation arising out of the stochastic differential equations has been integrated in the form of the path-integral which in turn has been approximately evaluated for the transition probability for the stationary case.

In a previous paper[4], we have generalized a deterministic linear model for *n*-species biological population, originally proposed by Coutlee *et al.*[2] and later by Gomatam[8] for two species interaction. There we have proposed the interactions

among the species, which follow Gompertz' law and discussed the situations where such models may be relevant. Also, the stochastic version of the model through the introduction of white noise type random components in the parameters of model has been investigated. The transition probability has been evaluated approximately under the steady state.

In the present paper we consider the generalized model, proposed in the above mentioned paper, for the situation when the growth and the interaction rates become time-dependent. Also we form the stochastic models from these deterministic equations and investigate the resulting stochastic differential equations. The corresponding Fokker-Planck equation has been solved approximately for the transition probability for the stationary case. In section 2, we introduce and investigate the model when some or all of the growth and the interaction rates are time-dependent. In the subsequent section 3, the stochastic model arising out of this deterministic model through the introduction of random components in the parameters, has been studied.

2. The Deterministic Model with Time-dependent Rates

The governing equations of the Gompertz-type-interaction model, which was proposed by the present author[4] in the previous paper to describe the interaction among the several biological species, are

$$\left.\begin{aligned} \frac{dy_i}{dt} &= \sum_j b_{ij} y_j + Z_i \qquad \text{for } t \geqslant 0 \\ \text{where} \quad y_i &= \ln X_i \qquad (i = 1, 2, .., n) \end{aligned}\right\} \qquad \ldots(1)$$

and

$$y_i = y_i(0) \text{ at } t = 0, \text{ for all } i \qquad \ldots(1a)$$

Z_i's $(i = 1, 2, \ldots, n)$ and b_{ij}'s $(i, j = 1, 2, \ldots, n)$

are the growth and the interaction rates respectively. X_i is the population density for the i th species at time t.

We now consider this deterministic population model with time-dependent growth and interaction rates. In this case, the governing equations are

$$\frac{dy_i}{dt} = \sum_j b_{ij}(t)\, y_j + Z_i(t) \qquad \ldots(2)$$

$$(i = 1, 2, \ldots, n)$$

with the initial conditions

$$y_i = y_i(0) \quad \text{at } t = 0. \qquad \ldots(2a)$$

Now let $b_{ij}(t)$ and $Z_i(t)$ be periodic functions of time with period τ. Such periodic nature of the growth and interaction rates may arise out of the seasonal changes in the environment or from periodic nature of mating of animals and subsequently the breeding of those species. Also, such fluctuations in the interaction terms occur due to periodic variations of common resources such as food, etc. Then setting $w_i(t;\ \tau) = y_i(t + \tau) - y_i(t)$; we have the following equation for $w_i(t;\ \tau)$:

$$\frac{dw_i(t;\ \tau)}{dt} = \sum_j b_{ij}(t)\, w_j(t;\ \tau). \qquad \ldots(3)$$

Let us specify the periodic functions $b_{ij}(t)$ and $Z_i(t)$ as,

$$\left.\begin{aligned} & b_{ij}(t) = (1 + \epsilon \cos pt)\, b_{ij} \\ \text{and} \quad & Z_i(t) = (1 + \epsilon \cos pt)\, Z_i \end{aligned}\right\} \qquad \ldots(4)$$

with $p = 2\pi/\tau$. b_{ij}'s and Z_i's are independent of time and ϵ is small.

Then we have from (2),

$$(1 + \epsilon \cos pt^{-1})\, \frac{d\bar{w}_i}{dt} = \sum_j b_{ij}\, \bar{w}_j \qquad \ldots(5)$$

where

$$\left.\begin{aligned} & \bar{w}_i = y_i + \alpha_i \\ \text{with} \quad & Z_i = \sum_j b_{ij}\, \alpha_j \end{aligned}\right\}. \qquad \ldots(6)$$

Here α_j's can be thought as the equilibrium values of y_i's for a model with time-independent growth and interaction rates. Using (4) we have from (3),

$$\frac{1}{1 + \epsilon \cos pt}\, \frac{dw_i(t;\ \tau)}{dt} = \sum_j b_{ij}\, w_j(t;\ \tau). \qquad \ldots(7)$$

The two set of equations (5) and (7) are of the same type. The solutions of them [for (5)] can be obtained by substituting

$$t + \epsilon\, \frac{\sin pt}{p} = T. \qquad \ldots(8)$$

Then eqns. (5) becomes,

$$\frac{d\bar{w}_i}{dT} = \sum_j b_{ij}\, \bar{w}_j. \qquad \ldots(9)$$

The solution of these equations are of the form :

$$\bar{w}_i = \sum_l A_l V_i^l\, e^{\lambda_l T} \qquad \ldots(10)$$

where V_i^l are the eigenfunctions of the matrix $\{b_{ij}\}$ with the eigenvalues λ_l and A_l are constants with respect to T. In the case of multiple roots of the characteristic or secular equation for the matrix $\{b_{ij}\}$, (that is, the case of some or all of the λ_i' s are equal, the above solutions are somewhat involved. In fact, in this case, the so-called similar terms appear and the constant eigenvectors V_i^l, in the formula, are to replaced by a polymomial in T, which are of the type

$$V_i^l + V_i^{\prime l} + \ldots.$$

Also, it should be noted that if the matrix $\{b_{ij}\}$ are symmetric, then the eigenvalues λ_l are all real, but in the general case λ_l may be complex. Now the solution becomes (for the case of λ_l 's are all the different)

$$\bar{\omega}_l = \sum_l A_l V_i^l \exp\left(\lambda_l\left(t + \epsilon \frac{\sin pt}{p}\right)\right) \qquad \ldots(11)$$

or

$$y_l = -\alpha_l + \sum_l A_l V_i^l \exp\left\{\lambda_l\left(t + \epsilon \frac{\sin pt}{p}\right)\right\}. \qquad \ldots(11a)$$

The constants A_l 's can be determined from the initial conditions (2a).

In the case when the interaction rates b_{ij} 's are constants whereas the growth rates $Z_i(t)$'s are perodic functions of time with period τ, we have from (2),

$$\frac{dy_l}{dt} = \sum_j b_{lj} y_j + Z_l(t). \qquad \ldots(12)$$

Integrating, from t to $t + \tau$, we have,

$$y_l(t + \tau) - y_l(t) = \sum_j b_{lj} \int_t^{t+\tau} (ty') \, dt' + \int_t^{t+\tau} Z_i(t') \, dt'. \qquad \ldots(13)$$

Now, using the solutions of (3) when b_{ij} 's are independent of time,

$$W_l(t;\ \tau) = y_l(t + \tau) - y_l(t) = \sum_l B_l V_i^l \ e^{\lambda}$$

we have, from (13),

$$\Sigma B_l V_i^l \; e^{\lambda_l t} = \sum_j b_{lj} \int_t^{t+\tau} y_j(t')\,dt' + \int_t^{t+\tau} Z_l(t')\,dt'.$$

From this it follows that,

$$\frac{1}{\tau}\int_t^{t+\tau} y_k(t')\,dt' = \frac{1}{\tau}\sum_{l,i} B_l\,(b^{-1})_{kl}\,V_i^l\; e^{\lambda_l t}$$

$$-\sum_l (b^{-1})_{kl}\,\frac{1}{\tau}\int_t^{t+\tau} Z_l(t')\,dt' \quad (k = 1, 2, \ldots, n) \qquad \ldots(14)$$

where $(b^{-1})_{kl}$ are the elements of the inverse matrix of the matrix $\{b_{ij}\}$.

These equations give the time averages over the period τ of the population sizes. When for all the eigenvalues λ_l. Re $\lambda_l < 0$, we have, as $t \to \infty$

$$<y_k>_\tau = -\sum_l (b^{-1})_{kl} <Z_l>_\tau$$

$$(k = 1, 2, \ldots, n) \qquad \ldots(15)$$

where $< \; >_\tau$ denotes the average over the period τ.

Now we proceed to solve (12) for the case when $Z_l(t)$ are arbitrary functions of time. Following Gantmacher[2], we use such a linear nonsingular transformation of variables

$$y = U\bar{v} \qquad (\det U \neq 0) \qquad \ldots(16)$$

that, $U^{-1}BU$ reduces to the "triangular" form,

$$U^{-1}\,B\,U = \begin{bmatrix} \lambda_1 & C_{12} & C_{13} & \vdots & C_{1n} \\ 0 & \lambda_2 & C_{23} & \vdots & C_{2n} \\ 0 & 0 & \lambda_3 & \vdots & C_{3n} \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & \vdots & \lambda_n \end{bmatrix} \equiv C\,(\text{say}) \qquad \ldots(17)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the matrix $\{b_{ij}\} \equiv B$, while the moduli of the nondiagonal elements C_{ik} $(i < k)$ may be made arbitrarily small by suitable choice of transformation (16).

By using this transformation, equation (12) reduces to

$$\left.\begin{aligned} \frac{d\bar{v}}{dt} &= C\,\bar{v} + \widetilde{Z}(t) \\ \text{where} \quad \widetilde{Z}(t) &= U^{-1} Z(t). \end{aligned}\right\} \qquad \ldots(18)$$

The last equation of this matrix equation is

$$\frac{d\bar{v}_n}{dt} = \lambda_n \bar{v}_n + \widetilde{Z}_n(t). \qquad \ldots(19)$$

Its solution is

$$\left.\begin{aligned} \bar{v}_n(t) &= e^{\lambda_n t} \int_0^t e^{-\lambda_n t}\, \widetilde{Z}_n(t)\, dt + e^{\lambda_n t}\, \bar{v}_n(0) \\ \text{where} \quad v(0) &= U^{-1} y(0). \end{aligned}\right\} \qquad \ldots(20)$$

Substituting $\bar{v}_n(t)$ from (20) into the last but one equation of (18), we get after solving for $\bar{v}_{n-1}(t)$,

$$\bar{v}_{n-1}(t) = e^{\lambda_{n-1} t} \int_0^t e^{-\lambda_{n-1} t}\, \widetilde{Z}_{n-1}(t) + e^{\lambda_{n-1} t}\, \bar{v}_{n-1}(0)$$

$$+ C_{n-1,n}\, e^{\lambda_{n-1} t} \int_0^t e^{-\lambda_{n-1} t + \lambda_n t} \{\int_0^t e^{-\lambda_n t}\, \widetilde{Z}_n(t)\, dt + \bar{v}_n(0)\} dt. \qquad \ldots(21)$$

In this way we can solve, exactly, all the equations in (18) for $\bar{v}_{n-2}(t)$, $\bar{v}_{n-3}(t)$, ..., $\bar{v}_1(t)$.

We now like to find the explicit solutions for a two-species population with time-dependent periodic and non-periodic growth rates. We take some specific values for the interaction coefficients, $b_{11} = -1$, $b_{21} = -1$, $b_{12} = 0.09$ and $b_{22} = 0$. The value of $b_{22} = 0$ corresponds to the zero value of the self-limiting term of the second species which helps to the growth of the first; whereas the first species is hostile to the second species. The U-matrix and its inverse, for this case, can be found to be respectively,

$$U = \begin{pmatrix} 0.1 & -\epsilon \\ 1 & 0 \end{pmatrix} \text{ and } U^{-1} = \begin{pmatrix} 0 & 1 \\ -1/\epsilon & 0.1/\epsilon \end{pmatrix} \qquad \ldots(22)$$

(ϵ Small)

Then matrix C becomes

$$C = U^{-1} BU = \begin{pmatrix} -0.1 & \epsilon \\ 0 & -0.9 \end{pmatrix} \qquad \ldots(23)$$

which is a triangular matrix with small off-diagonal element. Then eqns. (18) become

$$\frac{d}{dt}\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix} = \begin{pmatrix} -0.1 & \epsilon \\ 0 & -0.9 \end{pmatrix}\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix} + \begin{pmatrix} Z_2(t) \\ \dfrac{-Z_1(t) + 0.1\, Z_2(t)}{\epsilon} \end{pmatrix} \qquad \ldots(24)$$

where

$$\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \dfrac{-y_1 + 0.1\, y_2}{\epsilon} \end{pmatrix}.$$

We find the solutions of these equations for the first case of periodic growth rates :

$$Z_i(t) = \bar{p}_i(1 + \epsilon_i \cos pt) \; (i = 1, 2). \qquad \ldots(25)$$

The solutions are

$$\begin{aligned} y_1(t) = {} & 0.1\, e^{-0.1t}\, y_2(0) + \frac{-y_1(0) + 0.1 y_2(0)}{8}\,(e^{-0.1t} - 9\, e^{-0.9t}) \\ & + \bar{p}_2(1 - e^{-0.1t}) \\ & + \bar{p}_2\,\epsilon_2\, \frac{(p \sin pt + 0.1 \cos pt)}{0.1 + 10p^2} + (-\bar{p}_1 + 0.1\bar{p}_2) \\ & \times \left\{ \frac{(1 - e^{-0.1t})}{0.9} + \frac{(e^{-0.9t} - e^{-0.1t})}{7.2} \right\} \\ & + (-\bar{p}_1\,\epsilon_1 + 0.1\,\bar{p}_2\,\epsilon_2) \left\{ \frac{0.9\,(e^{-0.9t} - e^{-0.1t})}{8\,(0.81 + p^2)} \right. \\ & \left. + \frac{p \sin pt + (0.09 - p^2) \cos pt}{(0.81 + p^2)(0.1 + 10p^2)} \right\} \\ & + (\bar{p}_1 - 0.1\bar{p}_2)\, \frac{1 - e^{-0.9t}}{0.9} + (\bar{p}_1\,\epsilon_1 - 0.1\bar{p}_2\,\epsilon_2) \\ & \times \frac{(p \sin pt + 0.9 \cos pt) - 0.9 e^{-0.9t}}{0.81 + p^2} \end{aligned} \qquad \ldots(26a)$$

$$\begin{aligned} y_2(t) = {} & e^{-0.1t}\, y_2(0) + \frac{-y_1(0) + 0.1\, y_2(0)}{0.8}\,(e^{-0.1t} - e^{-0.9t}) \\ & + 10\bar{p}_2(1 - e^{-0.1t}) + \bar{p}_2\,\epsilon_2\, \frac{(p \sin pt + 0.1 \cos pt)}{0.01 + p^2} \\ & + (-\bar{p}_1\,\epsilon_1 + 0.1\bar{p}_2\,\epsilon_2) \left\{ \frac{9\,(e^{-0.9t} - e^{-0.1t})}{8\,(0.81 + p^2)} \right. \\ & \left. + \frac{(p \sin pt + (0.09 - p^2) \cos pt}{(0.81 + p^2)(0.01 + p^2)} \right\} \end{aligned}$$

(*equation continued on p.* 1088)

$$+ (-\bar{p}_1 + 0.1\,\bar{p}_2)\left\{\frac{1}{0.09}\,(1 - e^{-0\cdot 1t}) + \frac{1}{0.72}\,(e^{-0\cdot 9t} - e^{-0\cdot 1t})\right\}. \qquad \ldots(26b)$$

For the second case of non-periodic growth rate (a damping oscillatory) of the type

$$Z_i = \bar{p}_i + \epsilon_i\, e^{-at} \qquad (i = 1, 2) \qquad \ldots(27)$$

with a having positive real part, the solutions become

$$\begin{aligned} y_1(t) &= 0.1\, y_2(0)\, e^{-0\cdot 1t} + \frac{-y_1(0) + 0.1 y_2(0)}{8}\,(e^{-0\cdot 1t} - e^{-0\cdot 9t}) \\ &+ \frac{(-\bar{p}_1 + 0.1\bar{p}_2)}{9}\left\{\frac{1 - e^{-0\cdot 1t}}{0.1} + \frac{e^{-0\cdot 9t} - e^{-0\cdot 1t}}{0.8}\right\} \\ &+ \frac{(-\epsilon_1 + 0.1\,\epsilon_2)}{9 - 10a}\left\{\frac{e^{-at}}{0.1 - a} + \frac{e^{-0\cdot 9t}}{0.8}\right\} \\ &+ \bar{p}_2\,(1 - e^{-0\cdot 1t}) \\ &+ \epsilon_2\,\frac{e^{-at}}{1 - 10a} + (y_1(0) - 0.1 y_2(0))\, e^{-0\cdot 9t} \\ &+ \frac{(\bar{p}_1 - 0.1\bar{p}_2)}{0.9}\,(1 - e^{-0\cdot 9t}) \\ &+ \frac{\epsilon_1 - 0.1\,\epsilon_2}{0.9 - a}\,(e^{-at} - e^{-0\cdot 9t}) \end{aligned} \qquad \ldots(28a)$$

$$\begin{aligned} y_2(t) &= y_2(0)\, e^{-0\cdot 1t} + \frac{-y_1(0) + 0.1 y_2(0)}{0.8}\,(e^{-0\cdot 1t} - e^{-0\cdot 9t}) \\ &+ \frac{(-\bar{p}_1 + 0.1\bar{p}_2)}{0.9}\left\{\frac{1 - e^{-0\cdot 1t}}{0.1} + \frac{e^{-0\cdot 9t} - e^{-0\cdot 1t}}{0.8}\right\} \\ &+ \frac{(-\epsilon_1 + 0.1\,\epsilon_2)}{0.9 - a}\left\{\frac{e^{-at}}{0.1 - a} + \frac{e^{-0\cdot 9t}}{0.8}\right\} \\ &+ 10\,\bar{p}_2\,(1 - e^{-0\cdot 1t}) + \epsilon_2\,\frac{e^{-at}}{0.1 - a}. \end{aligned} \qquad \ldots(28b)$$

For large t, the behaviour of these solutions are, for the first case,

$$\left.\begin{aligned} y_1(t) &\approx \bar{p}_2\left\{1 + \epsilon_2\,\frac{p \sin pt + 0.1 \cos pt}{0.1 + 10p^2}\right\} \\ &\quad + (-\bar{p}_1\epsilon_1 + 0.1\bar{p}_2\,\epsilon_2)\left\{\frac{(0.9 - 10p^2)\, p \sin pt - 10p^2 \cos pt}{(0.81 + p^2)(0.1 + 10\,p^2)}\right\} \\ y_2(t) &\approx \frac{100}{9}\,(\bar{p}_2 - \bar{p}_1) + \bar{p}_2\,\epsilon_2\,\frac{p \sin pt + 0.1 \cos pt}{0.01 + p^2} \\ &\quad + (-\bar{p}_1\epsilon_1 + 0.1\,\bar{p}_2\,\epsilon_2)\left\{\frac{p \sin pt + (0.09 - p^2) \cos pt}{(0.81 + p^2)(0.01 + p^2)}\right\} \end{aligned}\right\} \qquad \ldots(29)$$

and for the second case,

$$\left.\begin{aligned} y_1(t) &\approx \bar{p}_2 \\ y_2(t) &\approx \frac{100}{9}\,(\bar{p}_2 - \bar{p}_1). \end{aligned}\right\} \quad \ldots(30)$$

Thus we see that, in the first case, the asymptotic behaviour of the soluations is periodic about the equilibrium solutions of the case with the time-independent parameters; whereas for the second case the solutions reach the equilibrium values asymptotically, that is, to the asymptotic stable values.

3. Discussion on Stochastic Generalization

In a previous paper[4], we have already made the stochastic generalization of the model with constant growth and interaction rates. Now we intend to generalize the present model by assuming that the interaction rates have random parts. That is we can write the governing equations as,

$$\frac{dy_i}{dt} = \sum_j (\bar{b}_{ij} + c_{ij}\,\gamma_j(t))\, y_j + Z_i(t) \quad \ldots(31)$$

where $\gamma_j(t)$ are δ-correlated Gaussian white noises,

with

$$\text{and} \quad \left.\begin{aligned} &<\gamma_j(t)> = 0 \\ &<\gamma_i(t+\tau)\;\gamma_j(t)> = \delta_{ij}\,\delta(\tau) \end{aligned}\right\} \quad \ldots(31\text{a})$$

and $\bar{b}_{ij}$'s are the averages of b_{ij}'s. We may write (31) as follows :

$$\left.\begin{aligned} &\dot{y}_i = K_i(y) + \sum_j g_{ij}(y)\;\gamma_j(t) \\ \text{with} \quad &K_i(y) = \sum_j \bar{b}_{ij}\, y_j + Z_i(t) \\ \text{and} \quad &g_{ij}(y) = C_{ij}\, y_j \end{aligned}\right\} \quad \ldots(32)$$

The corresponding Fokker-Planck equation can be written as[3,4]

$$\dot{P} = -\sum_{\mu} \frac{\partial}{\partial y_\mu}[A_\mu(y)\,P] + \frac{1}{2}\sum_{\mu,\nu} \frac{\partial^2}{\partial y_\mu\,\partial y_\nu}[B_{\mu\nu}(y)\,P] \quad \ldots(33)$$

with

$$\left.\begin{aligned} &A_\mu(y) = \sum_j (\bar{b}_{\mu j} + \tfrac{1}{2}\,C_{\mu j}\;C_{jj})\, y_j + Z_\mu(t) \\ \text{and} \quad &B_{\mu\nu}(y) = \sum_j C_{\mu j}\;C_{\nu j}\;y_j^2\,. \end{aligned}\right\} \quad \ldots(34)$$

The path integral solution of (33) can be found and the approximate expression for the the transition probability $P(y_l, T \mid y_i, 0)$ can be written following the same method of the previous papers[3,4]. They are, respectively,

$$\left.\begin{aligned} & P(y_l, T_l \mid 0) = \{\det B(y(T)\}^{-n/2} \int_{-\infty}^{\infty} D(y) \exp\{-\tfrac{1}{2}\int_0^T L\, dt\} \\ \text{where} \quad & L = [\dot{y} - \hat{A}(y)]^T B(y)^{-1} [\dot{y} - \hat{A}(y)] + 2 \sum_i C_{ii}^2 \\ & \hat{A}_l(y) = \sum_j (\bar{b}_{lj} - \frac{3}{2} C_{lj} C_{jj}) y_j + Z_l(t) \end{aligned}\right\} \quad \ldots(35)$$

and

$$P(y_l, T \mid y_i, 0) = \left\{\det B\left(y_i^c (T)\right)\right\}^{-n/2}$$

$$\exp\left\{-\frac{1}{2}\int_0^T L\left(y_i^c, \dot{y}_i^c\right) dt\right\} \int D(\xi) \exp\{-\tfrac{1}{2}\int 0\,(\xi^2)\, dt\} \quad \ldots(36)$$

where y_i^c 's are the solutions of the equations

$$\left.\begin{aligned} & \dot{y}_l = \sum_j q_{lj}\, y_j + Z_l(t), \quad (i = 1, 2, \ldots, n) \\ \text{with} \quad & q_{lj} = \bar{b}_{lj} - \tfrac{3}{2} C_{lj} C_{jj} \; (i, j = 1, 2. \ldots, n). \end{aligned}\right\} \quad \ldots(37)$$

The solutions of these equations can be obtained in the same way as for the equation (12) of the deterministic case and they are of the form (26) or (28) for the two-species populations, as for the example. From (36), we can then write down the expression for the transition probability using these solutions y_i^c 's.

For a particular case, when $C_{lj} = \epsilon_{lj} + C\,\delta_{lj}$ with ϵ_{ij}'s being very small compared to C; we find,

$$\left.\begin{aligned} & q_{lj} \approx \bar{b}_{lj} - \tfrac{3}{2}\{C^2 \delta_{lj} + C(\delta_{lj}\epsilon_{jj} + \epsilon_{lj})\} \\ \text{and} \quad & B_{\mu\nu}(y) \approx C^2 \delta_{\mu\nu}\, y_\nu^2 . \end{aligned}\right\} \quad \ldots(38)$$

In this case the transition probability becomes

$$P(y_l, \infty \mid y_i, 0) = D \exp\left\{-\frac{n}{2} \ln\left[\det B\left(y_i^c (\infty)\right)\right]\right\} \quad \ldots(39)$$

where D includes the path integral part for ξ's together with a constant term from $L\left((y_i^c , y_i^c \right)$, less important for the present discussions,

We have, using (38),

$$P(y_l, \infty \mid y_l, 0) = D \frac{1}{(C^{n^2/2})} \prod_{\mu} \frac{1}{\left(y_\mu^c (\infty) \right)^{-n}}. \qquad \ldots(40)$$

When $Z(t)$ is a periodic function with small amplitude, $y_\mu^c (\infty)$'s do not attain fixed values. On the contrary they oscillate with small amplitude around some fixed values. If $Z(t)$ is some slowly varying damping oscillatory function of time, y_μ^c 's can attain asymptotically stationary values. In this case the transition probability has some definite value.

In the present paper we have studied the deterministic models with time-dependent parameters. The methods of solutions have been discussed and the explicit solutions are also given for the two-species biological population. These models may also be relevant to other actual situations, more precisely to some 'Controlled' situations; as they are found to be relevant, for the cases of the time-independent parameters. discussed in the previous paper[4]. This may be investigated in future.

References

1. C. A. Braumann, *Bull. Math. Biol.*, **45** (1983), 635–41.
2. E. L. Coutlee, and R. I. Jennrich, *Amer. Naturalist*, **102** (1968), 307–21.
3. S. S. De, *Bull. Math. Biol.* **46** (1984), 175–84.
4. S. S. De, *Indian J. pure appl. Math.* **16** (1985), 1078–88.
5. F. Gantmacher, *Lectures in Analytical Mechanics*. Mir Publishers, Moscow, 1970.
6. N. S. Goel, S. C. Maitra, and E. W. Montroll, *Rev. Mod. Phys.* **43** (1971), 321–76.
7. N. S. Goel, and N. Richter-Dyn, *Stochastic Models in Biology*. Academic Press, 1974.
8. J. Gomatam, *Bull. Math. Biol.* **36** (1974), 347–53.
9. R. M. May, *Stability and Complexity in Model Ecosystems*. Princeton University Press, 1973.

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NOTE ON A THEOREM OF ALTMAN

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The purpose of the present study is to clarify and Generalize a theorem of Altman[2].

Let Q be a real-valued function satisfying

(a) $0 < Q(s) < s$ for $s < 0$ and $Q(0) = 0$,

(b) $g(s) := s/(s - Q(s))$ is nonincreasing on $(0, \infty)$,

(c) $\int_0^{s_1} g(s)\, ds < \infty$ for each $s_1 > 0$, and

(d) $Q(s)$ is nondecreasing.

We suppose throughout that X is a complete metric space. A function $F : X \to X$ is called a generalized contraction if $d(Fx, Fy) \leqslant Q(d(x, y))$ for all $x, y \in X$. Altman[1] has shown that functions Q, having all four properties above, arise naturally in a Banach space setting.

The purpose of this note is to clarify and generalize the following theorem of Altman[2]:

Theorem 1—A generalized contraction has a unique fixed point z and, for every $x_0 \in X$, $\lim_{n \to \infty} (F)^n (x_0) = z$.

Actually, Altman's version has the following condition on Q replacing (a).
(a^1) $0 < Q(s) < s$ for $0 < s \leqslant s_1$.

This form does not give uniqueness since s_1 depends on the starting point x_0. However, (a) is sufficient for uniqueness and the rest of the proof goes through as in Altman[2].

We generalize Theorem 1 for a pair of mappings, one of which is a generalized contraction and the other expansive.

Theorem 2—Let $F, G : X \to X$ be such that

(1) F is a generalized contraction,

(2) $d(Gx, Gy) \geqslant d(x, y)$ for all $x, y \in X$, and

(3) $F(X) \subseteq G(X)$.

Then $F(x) = G(x)$ has a unique solution z and, for every $x_0 \in X$, $\lim_{n \to \infty} (G^{-1} F)^n (x_0) = z$.

PROOF : First, G^{-1} exists since (2) implies G is one-to-one. Also, (3) gives $G^{-1} F : X \to X$. Next it follows from (2) that

$$d(GG^{-1} x, GG^{-1} y) > d(G^{-1} x, G^{-1} y)$$

or

$$d(x, y) \geqslant d(G^{-1} x, G^{-1} y).$$

Replacing x by Fx and y by Fy in the latter inequality gives

$$d(G^{-1} Fx, G^{-1} Fy) \leqslant d(Fx, Fy) \leqslant Q(d(x, y)).$$

Hence $G^{-1} F$ is a generalized contraction with respect to Q. The result now follows by Theorem 1.

We note that taking G to be the identity map in Theorem 2 gives Altman's result.

In order to illustrate Theorem 2, we give the following example. Let $X = [0, \infty)$ and let d be the absolute value metric. Define $F, G : [0, \infty) \to [0, \infty)$ by $F(x) = \frac{1}{4 + x}$ and $G(x) = 2x$. Let $Q(s) = \frac{s}{s + 16}$. Clearly Q satisfies conditions (a), (b), (c) and (d). Then $d(Fx, Fy) = \frac{|x - y|}{xy + 4(x + y) + 16} \leqslant \frac{|x - y|}{|x - y| + 16} = Q(d(x, y))$ and $d(Gx, Gy) = 2|x - y| \geqslant |x - y|$. Also $FX = \left(0, \frac{1}{4}\right] \subseteq [0, \infty) = GX$.

Hence, all conditions of Theorem 2 are fulfilled and $2x = \frac{1}{x + 4}$ has a unique solution in $[0, \infty)$.

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REFERENCES

1. M. Altman, *Bull. Austr. Math. Soc.* 10 (1974), 51–58,
2. M. Altman, *Amer. Math. Monthly* 82 (1975), 827–29.

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SOME RESULTS IN THE FIXED POINT THEORY, III

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In this paper we give applications of our Localization Monotone Principle in the fixed point theory.

Let X be a topological space, $T : X \to X$ and let $B : X \to \mathbb{R}^0_+ : =[0, \infty)$ be a T-orbitally lower semicontinuous function on X. A function B mapping X into the reals is T-orbitally lower semicontinuous at p if $\{x_n\}$ is a sequence in $O(x) : = \{x, Tx, T^2 x, \ldots\}$, $x \in X$ and $x_n \to p$ implies that $B(p) \leqslant \liminf B(x_n)$. A space X is said to be T-orbitally complete if every Cauchy sequence which is contained in $O(x)$ for some $x \in X$ converges in X (c. f. Tasković[14]).

In connection with this, in a recent paper[14] we introduced the concept of TCS-convergence in a space X; i. e., a topological space X satisfies the condition of TCS-convergence if there exists a point $x \in X$ such that $B(T^n x) \to 0$ $(n \to \infty)$ implies $\{T^n x\}$ has a convergent subsequence. We stated a new fixed point principle :

Localization Monotone Principle[14]—Let T be a mapping of a topological space X into itself, where X satisfies the condition of TCS-convergence. Suppose that there exists $y \in X$ and a mapping $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ such that $\varphi(t) < t$ and $\limsup_{z \to t+0} \varphi(z) < t$ for every $t > 0$ and

$$B(Tx) \leqq \varphi(B(x)), \text{ for every } x \in O(y) \qquad \ldots(1)$$

where $B : X \to \mathbb{R}$ is T-orbitally lower semicontinuous and $B(x) = 0$ implies $Tx = x$. Then T has a unique fixed point $\xi \in X$.

We get the following from the preceding principle :

Corollary 1—Let T be a mapping of a topological space X into itself. Suppose that there exist $\alpha \in [0, 1)$ and

$$B(Tx) \leqq \alpha B(x), \text{ for all } x \in X,$$

where $B : X \to \mathbb{R}^0_+$ is T-orbitally lower semicontinuous and $B(x) = 0$ implies $Tx = x$. If for some $x \in X$ the sequence $\{T^n x\}$ has a convergent subsequence, then T has a fixed point $\xi \in X$.

Corollary 2—Let T be a mapping of metric space (X, ρ) into itself and let X be T-orbitally complete. Suppose that there exist $\alpha \in [0, 1)$ and

$$A(Tx, Ty) \leqslant \alpha A(x, y), \text{ for all } x, y \in X$$

where $A : X \times X \to \mathbb{R}^0_+$, $x \to A(x, Tx)$ is lower semicontinuous and $\rho[x, y] \leqslant A(x, y)$ for all $x, y \in X$. Then T has a unique fixed point $\xi \in X$ and $T^n x \to \xi$ for each $x \in X$.

We notice, this result is generalization of the results of Chakrabarty[2].

In this paper, we apply our Localization Monotone Principle in the fixed point theory.

In this part we assume that (X, ρ) is a complete metric space and that Φ is a mapping of $\mathbb{R}^0_+$ into itself satisfying the conditions : Φ is nondecreasing and continuous on the right, $\Phi(t) < t$ for all $t > 0$. Further we assume that Ψ is a continuous mapping of $X \times X$ into $\mathbb{R}^0_+$ satisfying the conditions $\Psi(x, x) = 0$ and $\rho(x, y) \leqslant \Psi(x, y)$, for any x, y in X.

Corollary 3 (Chakrabarty[2])—Let M be a subset of X and T be a mapping of M into itself. Suppose that

$$\Psi(Tx, Ty) \leqslant \Phi(\Psi(x, y)), \text{ for all } x, y \in M \qquad \ldots(2)$$

where $\Psi(x, T^n x) \leqslant A(x)$, $n \in \mathbb{N}$ for every $x \in M$ and $A(x)$ a positive number. Then there is a point ξ in X such that for any $x \in X$, $T^n x \to \xi$ $(n \to \infty)$. If M is a closed, then ξ is the unique fixed point of T.

PROOF : Let $B(x) = \Psi(x, Tx)$ and $\varphi(t) = \Phi(t)$. It is easy to see that B and φ satisfy all the required hypotheses in localization monotone principle. Also, from (2) X satisfies the condition of TCS-convergence, because X is a complete metric space. This completes the proof.

Popa[9] obtained a fixed point theorem which is a common generalization of results of Banach, Ray[10] and Jaggi[7]. In this paper we extend this statement. Namely, we give applications of localization mono'one principle, and as special case results of Popa[9], also and some others results.

Corollary 4 (Popa[9])—Let T be a continuous mapping of a Hausdorff spaces X into itself and let f be a continuous mapping of $X \times X$ into the nonnegative reals such that for some $a, b \in \mathbb{R}_+$, $a + b < 1$ and for all $x, y \in X$ and $x \neq y$:

$$f(Tx, Ty) \leqslant a f(x, Tx) f(y, Ty) (f(x, y))^{-1} + b f(x, y) \qquad \ldots(3)$$

and

$$f(x, x) f(y, y) \leqslant f^2(x, y) \text{ and } f(x, y) \neq 0. \qquad \ldots(4)$$

If for some $x \in X$ the sequence of iterates $\{T^n x\}$ has a convergent subsequence, then T has a unique fixed point.

PROOF: Let x be an arbitrary point in X. Then, for $y = Tx \neq x$, from (3) we have

$$f(Tx, T^2x) \leqslant (a+b) f(x, Tx), \text{ for all } x \in X.$$

Let $B(x) = f(x, Tx)$ which is lower semicontinuous on X, and let $\varphi(t) = (a+b)\,t$, for $t \in \mathbf{R}_+$; then B and φ satisfy all the required hypotheses in our localization monotone principle. Since X satisfies the condition of TCS-convergence (the sequence of iterates $\{T^n x\}$ has a convergent subsequence), applying our localization monotone principle we obtain $T\xi = \xi$ for some $\xi \in X$. Uniqueness follows immediately from conditions (3) and (4).

Corollary 5 (Jaggi[7], Bose and Mukherjee[8])—Let T be a mapping of a metric space X into itself satisfying the following condition

$$\rho[Tx, Ty] \leqslant a\, \rho[x, Tx]\, \rho[y, Ty]\, (\rho(x, y))^{-1} + b\, \rho[x, y],\ x \neq y$$

where a and b are nonnegative real numbers such that $a + b < 1$. If there exists an orbit $O(x_0)$ which contains a convergent subsequence of which T is orbitally continuous, then T has a unique fixed point.

PROOF: is analogous to the proof of the preceding statement of Popa[9].

We note, this statement of Bose and Mukherjee[1] is special case of Corollay 3 of Popa[9].

Let $\mathbb{N}$ denote the set of all nonnegative integers. A pair $(X, \to)$ of a set X and a subset $\to$ of the set $X^{\mathbb{N}} \times X$ is called an L-space if the following two conditions are satisfied:

(i) If $x_n = x \in X$ for all $n \in \mathbb{N}$, then $(\{x_n\}, x) \in \to$.

(ii) If $(s, x) \in \to$, then $(t, x) \in \to$ for every subsequence t of s.

In what follows, we shall write $s \to x$ or $x_n \to x$ instead of $(s, x) \in \to$, and read s converges to x, where $s = \{x_n\}, n \in \mathbb{N}$. If $s = \{x_n\}, n \in \mathbb{N}$, is a sequence in a set X, and if f is a mapping on X, then $f(s)$ denotes the sequence $\{f(x_n)\}$. Let $(X, \to)$ be an L-space. It is said to be separated if each sequence in X converges to at most one point of X. Let d be a nonnegative extended real valued function on $X \times X$. The L-space $(X, \to)$ is said to be d-complete if each sequence $\{x_n\}_{n \in \mathbb{N}}$ in X with $\sum_{n=0}^{\infty} d(x_{n-1}, x_n) < \infty$ converges to at least one point of X.

The following statement is corollary of our localization monotone principle:

Corollary 6 (Kasahara[6])—Let $(X, \rightarrow)$ be a separated L-space which is d-complete for a nonnegative extended real valued function d on $X \times X$, and f be a continuous mapping of X into itself satisfying the following conditions for some α, β with $0 \leqslant \alpha < 1$ and $0 < \beta \leqslant \infty$:

$$d(f(x), f^2(x)) \leqslant \alpha d(x, f(x)) < \alpha\beta, \text{ for all } x \in X. \qquad \ldots(5)$$

Then f has a fixed point, and the sequence $\{T^n x\}$ converges to a fixed point of f.

PROOF : As can readily be seen by induction, $d(f^n(a), f^{n+1}(a) \leqslant \alpha^n d(a, f(a))$, for every $n \in \mathbb{N}$, where $\mathbb{N}$ denote the set of all nonnegative integers. Hence the d-completeness of the space implies that the sequence $\{f^n x\}$ converges to some $\xi \in X$. This implies that X satisfies the condition of TCS-convergence. Let $B(x) = d(x, f(x))$ and $\varphi(t) = \alpha t$. Applying localization monotone principle we get $\xi = f\xi$ for some $\xi \in X$.

Also, an immediate corollary of the our preceding localization monotone principle is :

Corollary 7—Let T be a mapping of metric space (X, ρ) into itself. Suppose that there exists $\alpha \in [0, 1)$ such that

$$\rho[Tx, T^2x] \leqslant \alpha\rho[x, Tx], \text{ for all } x \in X$$

where $x \rightarrow \rho[x, Tx]$ is T-orbitally lower semicontinuous and $\{T^n x\}$ has a convergent subsequence. Then T has a fixed point in X.

Corollary 8 (Dhage[3])—Let $T : X \rightarrow X$ be an orbitally continuous self-map of a metric space X and let X be T-orbitally complete. If T satisfies the condition

$$\min\{\rho[Tx, Ty], \rho[x, Tx], \rho[y, Ty]\}$$

$$+ a \min\{\rho[x, Ty], \rho[y, TX], \leqslant p\,\rho[x, y] + q\rho[x, Tx] \qquad \ldots(6)$$

for all $x, y \in X$ and a, p and q are real numbers such that $0 < p + q < 1$, then for each $x \in X$, then sequence $\{T^n x\}$ converges to a fixed point of T.

PROOF : Let $x \in X$ be an arbitrary point in X. Then, for $y = Tx$, from (6) we have

$$\min\{\rho[Tx, T^2x], \rho[x, Tx]\} \leqslant (p + q)\,\rho[x, Tx].$$

Hence, for $B(x) = \rho[x, Tx]$, $\varphi(t) = (p + q)\,t$, and since X satisfies the condition of TCS-convergence (X is T-orbitally complete metric space and $\rho[T^n x, T^{n+s}x] \leqslant (p+q)^n (1 - p - q)^{-1}\rho[x, Tx]$), applying our localization monotone principle we obtain $T\xi = \xi$ for some $\xi \in X$.

Some Remarks

(1) We notice, that and some other results (Theorems 2, 3 and 4) of Dhage[3], are special case of our localization monotone principle. Proofs are analogous of the preceding proof of Corollary 4.

(2) Also, in a paper[13] we have proved the following statement, which generalizes the preceding statement of Dhage[3]. A comparative study of these generalization has been made more recently by Yeh[15] and Khan[8].

Corollary 9 (Tasković[13])—Let $T : X \to X$ be a mapping on X and let X be a T-orbitally complete metric space. If there exists real numbers α_i, β for every $x, y \in X$ such that $\alpha_1 + \alpha_2 + \alpha_3 > \beta$ and $\beta - \alpha_2 \geqq 0$ or $\beta - \alpha_3 \geqq 0$, and

$$\alpha_1 \rho [Tx, Ty] + \alpha_2 \rho [x, Tx] + \alpha_3 \rho [y, Ty] + \alpha_4 \min \{\rho [x, Ty], \rho [y, Tx]\} \leqslant \beta\rho [x, y] \qquad \ldots(7)$$

then for each $x \in X$, then sequence $\{T^n x\}$ converges to a fixed point ξ of T.

Proof is analogous on the proof of the Corollary 6.

Also, we note, Corollary 6 of Dhage[3] which the assumption of the continuity of T is removed. Also, the condition (6) implies out the condition (7).

Let X denote a metric space and K a bounded subset of X. Following Kuratowski[5] we denote by $\alpha(K)$ the infimum of all $\epsilon > 0$ such that K admits a finite converging with subset of diameters less that ϵ. We use the following properties of the number α.

1. $\alpha(K) = 0$ if and only if K is precompact. For this reason $\alpha(K)$ is called the measure of noncompactness of K.

2. $\alpha(A \cup B) = \max \{\alpha(A), \alpha(B)\}$, $(A, B \subset X)$.

3. If K is compact then $\alpha(K) = 0$. Also, $\alpha(C1K) = 0 \Leftrightarrow \alpha(K) = 0$, and $0 \leqq \alpha(K) \leqq \delta(K)$.

4. $\alpha(K) = 0$ and X complete imply K is compact.

We list below some of contractive mappings for which various fixed point theorems have been stablished :

(1) (Furi and Vignoli[4]). The continuous mapping T is called densifying, if for every bounded subset A of X, such that $\alpha(A) > 0$, we have $\alpha(T(A)) < \alpha(A)$. Let F be a real lower semicontinuous function defined in $X \times X$. The densifying mapping T is said to be weakly F-contractive if the condition $F(Tx, Ty) < F(x, y)$ holds for all $x, y \in X$, $(x \neq y)$.

(2) (Ray and Chatterjee[11], Singh[12], Khan[8]). Let $F : X \times X \to R_{+}^{0}$ be continuous and $T : X \to X$ be densifying mapping such that $F(Tx, Ty) < \alpha F(x, y) + \beta F(x, Tx) + \gamma F(y, Ty)$ for each pair, of distinct points $x, y \in X$ and for nonnegative real numbers α, β, γ with $\alpha + \beta + \gamma < 1$, then T is called generalized densifying.

(3) (Bose and Mukherjee[1]) Let F be a continuous symmetric mapping of $X \times X$ into the set of nonnegative reals, such that $F(x, y) = 0$ iff $x = y$ and

$$F(Tx, Ty) \leqslant aF(x, Tx)\, F(y, Ty)\, (F(x, y))^{-1} + bF(x, y),\ x \neq y$$

for each pair of distinct points $x, y \in X$ where $a + b < 1$ $(a, b \geqslant 0)$.

Corollary 10 (Furi and Vignoli[4])—Let $T : X \to X$ be a densifying (therefore, by definition, continuous) and weakly F-contractive mapping defined in a complete metric space. If for some $x_0 \in X$, the sequence $\{x_n\}$ of iterates starting from $x_0 = x$ is bounded, then T has a unique fixed point ξ in X.

PROOF : For $O(x)$ we have $TO(x) \subset O(x)$, then $O(x)$ is an invariant set. Denote by $\mathrm{Cl}O(x)$ the closure of $O(x)$. $\mathrm{Cl}O(x)$ is an invariant set, too; indeed, from the continuity of T it follows $T(\mathrm{Cl}\, O(x)) \subset \mathrm{Cl}\, T(O(x)) \subset \mathrm{Cl}\, O(x)$. Now we shall prove that $\mathrm{Cl}O(x)$ is compact. For this it is sufficient to show that $\alpha(O(x)) = 0$, since in a complete metric space the precompact sets are also relatively compact. Suppose $\alpha(O(x)) > 0$; in this case

$$\alpha(T(O(x))) < \alpha(O(x)) \text{ and since } O(x) = TO(x) \cup \{x\} \qquad \ldots(8)$$

it follows that

$$\alpha(O(x)) = \max\{\alpha(TO(x)), \alpha(x)\} + \max\{\alpha(TO(x)), 0\} = \alpha(TO(x)).$$

But this contradicts (8), hence $\alpha(O(x)) = 0$;

Also, let $B(x) = F(x, Tx)$. Since T is continuous and F is lower semicontinuous, B is lower semicontinuous, it is easy see to that B: $X \to R_{+}^{0}$ satisfies all the required hypothesis in localization monotone principle. Hence, applying the localization monotone principle we obtain $T\xi = \xi$, for some $\xi \in X$. Uniqueness, it follows immediately from the weak F-contractivity.

Analogous to the proof for the preceding statement we have the same proof for the statements of Ray and Chatterjee[11], Singh[12], Khan[8], Bose and Mukherjee[1].

REFERENCES

1. R. Bose, and R. Mukherjee, *Prog. Math.* 16 (1982), 1–7.
2. B. Charkrabarty, *Math. Balkanica*, 8 (1978), 27–41.
3. B. Dhage, *Indian J. pure appl. Math.* 16 (1985), 245–56.
4. M. Furi, and A. Vignoli, *Boll. U.M.I.* 4 (1969), 505–509.
5. C. Kuratowski, *Topologic.* Monographe Mathematizne, 20 (1958), Warszawa.

6. S. Kasahara, *Publ. RIMS*, **12** (1976), 427–37.
7. S. Jaggi, *Indian J. pure appl. Math.* **8** (1977), 230–33.
8. M. S. Khan, *Publ. Inst. Math.* **27** (1980), 107–112.
9. V. Popa, *Indian J. pure appl. Math.* **14** (1983), 713–17.
10. B. K. Ray, *Bull. Math. Sci. Math. de la R.S.R.* **16** (1972), 467–69.
11. B. Ray, and H. Chatterjee, *Bull. Acad. Pol. Sci.* **25** (1977), 1243–47.
12. S. P. Singh, *Rend. Semi. Math. Univ. Padova* **43** (1970), 229–32.
13. M. Tasković, *Publ. Inst. Math.* **20** (1976), 231–42.
14. M. Tasković, *Proc. Am. Math. Soc.* **94** (1985), 427–32.
15. C. Yeh, *Indian J. pure appl. Math.* **9** (1978), 993–95.

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ON FEEBLY CLOSED MAPPINGS

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Recently the notion of feebly closed mappings was introduced. In this paper this concept is shown to coincide with the notion of α-closed mappings. Furthermore it is shown that if the codomain is appropriately retopologised the concept of feebly closed mappings coincides with the usual notion of closed mappings. Some properties of α-closed mappings are investigated.

1. INTRODUCTION

Let S be a subset of topological space $(X, \mathscr{T})$. We denote the closure of S and the interior of S with respect to $\mathscr{T}$ by $\mathscr{T}$ cl S and $\mathscr{T}$ int S respectively, although we may suppress the $\mathscr{T}$ when there is no possibility of confusion. We denote the topology induced by $\mathscr{T}$ on S, by $\mathscr{T}_S$. Njåstad[8] introduced the concept of an α-set in $(X, \mathscr{T})$. A subset S of $(X, \mathscr{T})$ is called an α-set if $S \subset \mathscr{T}$ (int $\mathscr{T}$ cl ($\mathscr{T}$ int S)). The notions of semi-open set and preopen set were introduced by Levine[3] and Mashhour *et al.*[5] respectively. A subset S of $(X, \mathscr{T})$ is called a semi-open set (respectively preopen set) if $S \subset \mathscr{T}$ cl ($\mathscr{T}$ int S) (respectively $S \subset \mathscr{T}$ int ($\mathscr{T}$ cl S)). The complements of an α-set, a semi-open set and a preopen set are called α-closed, semi-closed and preclosed respectively. Following Njåstad[8] we denote the family of all α-sets in $(X, \mathscr{T})$ by $\mathscr{T}^\alpha$. Njåstad proved that $\mathscr{T}^\alpha$ is a topology on X. As any open set in $(X, \mathscr{T})$ is an α-set, $\mathscr{T} \subset \mathscr{T}^\alpha$ in the lattice of topologies on the set X. If A is a subset of $(X, \mathscr{T})$, then the intersection of all semiclosed sets containing A is called the semi closure of A, and is denoted s cl A. The largest semi-open set contained in A is denoted by s int A. Maheshwari and Tapi[7] defined A to be a feebly open set in $(X, \mathscr{T})$ if there is an open set U such that $U \subset A \subset s$ cl U. The complement of a feebly open set is called a feebly closed set.

2. FEEBLY CLOSED MAPS

The concepts of α-closed and feebly closed mappings have been introduced by Mashhour *et al.*[6] and Maheshwari and Jain respectively.

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Definition 1—A function $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is called

(i) feebly closed if the image of each closed set in X is feebly closed in Y;

(ii) α-closed if the image of each closed set in X is α-closed in Y.

Lemma 1—Let A be a subset of $(X, \mathcal{T})$. Then $s\text{int}\,(\text{cl}\,A) = \text{cl}\,(\text{int}\,(\text{cl}\,A))$.

PROOF : Notice that cl (int (cl A)) is a semi-open set since $\text{cl}\,(\text{int}\,(\text{cl}\,A)) = \text{cl}\,(\text{int}\,(\text{cl}\,(\text{int}\,(\text{cl}\,A))))$, and $\text{cl}\,(\text{int}\,(\text{cl}\,A)) \subset \text{cl}\,A$. Therefore $\text{cl}\,(\text{int}\,\text{Cl}\,A)) \subset s\,\text{int}\,(\text{cl}\,A)$.

Conversely, if U is any semi-open set contained in cl A, then $U \subset \text{cl}\,(\text{int}\,U) \subset \text{cl}\,(\text{int}\,(\text{cl}\,A))$ and therefore $s\,\text{int}\,(\text{cl}\,A) \subset \text{cl}\,(\text{int}\,(\text{cl}\,A))$.

Maheshwari and Jain[6] in Lemma 3 of their paper showed that a subset A of $(X, \mathcal{T})$ is feebly closed if and only if $s\,\text{int}\,(\text{cl}\,A) \subset A$.

Proposition 1—If A is a subset of $(X, \mathcal{T})$, then A is feebly closed if and only if A is α-closed.

PROOF : It follows from the definitions of an α-set and an α-closed set that a subset A of $(X, \mathcal{T})$ is α-closed if and only if $\text{cl}\,(\text{int}\,(\text{cl}\,A)) \subset A$.
Since $\text{cl}\,(\text{int}\,(\text{cl}\,A)) \subset A$ if and only if $s\,\text{int}\,(\text{cl}\,A) \subset A$, by Lemma 1, A is α-closed if and only if A is feebly closed.

An alternative proof of this result has been given in Proposition 1 of Janković and Reilly[2], who proved that if A is a subset of $(X, \mathcal{T})$, then A is feebly open if and only if A is an α-set.

Proposition 2 follows immediately from Proposition 1 and Definition 1.

Proposition 2—The following are equivalent :

(1) $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is feebly closed;

(2) $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed;

(3) $f: (X, \mathcal{T}) \to (Y, \mathcal{U}^\alpha)$ is closed.

If the codomain space of a feebly closed mapping f is retopologised in an obvious way, then f is simply a closed mapping. This observation puts the notion of feebly closed mappings into a more natural setting, and enables us to provide immediate proofs of some of the results in Maheshwari and Jain[6]. For example, Propositions 5 and 6 of Maheshwari and Jain[6] are well-known results (Murdeshwar[4], Theorem 4.26, P. 96 and Theorem 4.28, p. 26) restated in this setting.

3. α-CLOSED MAPPINGS

The following classes of generalized closed mappings were introduced in Mashhour *et al.*[5] and Noiri[9].

Definition 2—A function $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is called

(i) semi-closed if the image of each closed set in X is semi-closed in Y;

(ii) preclosed if the image of each closed set in X is preclosed in Y.

It is shown in Theorem 3 of Reilly and Vamanamurthy[10] that a subset of $(X, \mathcal{T})$ is an α-set if and only if it is semi-open and preopen. Thus we have the following result.

Proposition 3—A function $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed if and only if it is semi-closed and preclosed.

Examples 1 and 2 show that the separate converses are not in general true.

Example 1—Let $X = \{a, b, c\}$ and define the topologies $\mathcal{T}$ to be the discrete topology and $\mathcal{U} = \{\phi, x, \{a\}, \{c\}, \{a, c\}\}$. We define $f : (X, \mathcal{T}) \to (X, \mathcal{U})$ by $f(a) = f(b) = f(c) = a$. Then f is preclosed but not α-closed since $\{a\}$ is preclosed in $(X, \mathcal{U})$ but not α-closed in $(X, \mathcal{U})$.

Example 2—If $\mathcal{T}$ is the discrete topology and $\mathcal{U}$ is the indiscrete topology then $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is semi-closed but not α-closed.

Andrijević[1] showed that if M is a subset of $(X, \mathcal{T})$ then $(\mathcal{T}_M)^\alpha \subset (\mathcal{T}^\alpha)_M$ (his Theorem 3.2) and if M is preopen then $(\mathcal{T}_M)^\alpha = (\mathcal{T}^\alpha)_M$ (his Theorem 3.6).

Proposition 4—Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ and $f(X) \subset Y_1 \subset Y$.

(1) If $f : (X, \mathcal{T}) \to (Y_1, \mathcal{U}_{Y_1})$ is α-closed, then so is $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$.

(2) If Y_1 is preopen and $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed, then $f : (X, \mathcal{T}) \to (Y_1, \mathcal{U}_{Y_1})$ is α-closed.

PROOF : (1) If $f : (X, \mathcal{T}) \to (Y_1, \mathcal{U}_{Y_1})$ is α-closed then $f : (X, \mathcal{T}) \to (Y_1, (\mathcal{U}_{Y_1})^\alpha)$ is closed. By Andrijević's result $f : (X, \mathcal{T}) \to (Y_1, (\mathcal{U}^\alpha)_{Y_1})$ is closed and by Theorem 4.24 (2) of Murdeshwar[4] $f : (X, \mathcal{T}) \to (Y, \mathcal{U}^\alpha)$ is closed. Hence $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed.

(2) If $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed then $f : (X, \mathcal{T}) \to (Y, \mathcal{U}^\alpha)$ is closed and by Theorem 4.24, of Murdeshwar,[4] $f : (X, \mathcal{T}) \to (Y_1, (\mathcal{U}^\alpha)_{Y_1})$ is closed. If Y_1 is preopen then by Andrijević's result $f : (X, \mathcal{T}) \to (Y_1, (\mathcal{U}_{Y_1})^\alpha)$ is closed, and so $f : (X, \mathcal{T}) \to (Y_1, \mathcal{U}_{Y_1})$ is α-closed.

Proposition 5—If f is an α-closed mapping $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$, $\mathcal{T}_1 \subset \mathcal{T}$ and $\mathcal{U}^\alpha \subset \mathcal{U}_1^\alpha$ then $f : (X, \mathcal{T}_1) \to (Y, \mathcal{U}_1)$ is α-closed.

PROOF : If $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed then $f: (X, \mathcal{T}) \to (Y, \mathcal{U}^{\alpha})$ is closed, and by Theorem 4.23 of Murdeshwar[4], $f: (X, \mathcal{T}_1) \to (Y, \mathcal{U}_1^{\alpha})$ is closed. Hence $f: (X, \mathcal{T}_1) \to (Y, \mathcal{U}_1)$ is α-closed.

Proposition 6—If $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is an α-closed mapping and $B, C \subset Y$, then if $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint neighbourhoods, B and C have disjoint neighbourhoods in $(Y, \mathcal{U}_{\alpha})$.

PROOF : If $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed then $f: (X, \mathcal{T}) \to (Y, \mathcal{U}^{\alpha})$ is closed and the result follows from Murdeshwar[4] Theorem 4.28 (2).

Proposition 7—Let $f: X \to (Y, \mathcal{U})$ and let X be given the preimage topology $\mathcal{T}$. Then f is α-closed if and only if $f(X)$ is α-closed in $(Y, \mathcal{U})$.

PROOF : One implication, namely if f is α-closed then $f(X)$ is α-closed in $(Y, \mathcal{U})$ is clear since X is closed in the preimage topology.

Conversely, if $f(X)$ is α-closed in $(Y, \mathcal{U})$ then $f(X)$ is closed in $(Y, \mathcal{U}^{\alpha})$. Let $\mathcal{T}_{\alpha}$ be a preimage topology on X for $f: X \to (Y, \mathcal{U}^{\alpha})$.

Then, by Theorem 4.30 of Murdeshwar[4] $f: (X, \mathcal{T}_{\alpha}) \to (Y, \mathcal{U}^{\alpha})$ is closed. Since $\mathcal{U} \subset \mathcal{U}^{\alpha}$ and therefore $\mathcal{T} \subset \mathcal{T}_{\alpha}$, $f: (X, \mathcal{T}) \to (Y, \mathcal{U}^{\alpha})$ is closed, and so $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed.

It is well known that the T_1 property is preserved under closed mappings. The following example shows that the T_1 property is not preserved under α-closed mappings.

Example 3—Let X be an infinite set and p be a fixed point of X. We define a topology $\mathcal{T}$ on X as follows : for $G \subset X, G \in \mathcal{T}$ if $G = \phi$ or $G = X$ or $X - G$ is finite. We define a topology $\mathcal{U}$ on X as follows : for each $G \subset X, G \in \mathcal{U}$ if (i) $G = \phi$ or $G = X$, or (ii) $G \subset X - \{p\}$ and $X - G$ is finite. $(X, \mathcal{T})$ is T_1 but $(X, \mathcal{U})$ is not T_1 since for any point x distinct from p the only open set containing p, namely X, contains x. Let $f: (X, \mathcal{T}) \to (X, \mathcal{U})$ be the identity function. Then f is α-closed since if A is a closed subset of $(X, \mathcal{T})$ then either A is a closed subset of $(X, \mathcal{U})$ and therefore α-closed in $(X, \mathcal{U})$, or A is finite, nonempty and $p \notin A$. In this case $\mathcal{U}$ cl ($\mathcal{U}$ int ($\mathcal{U}$ cl A)) $= \phi \subset A$ and therefore A is an α-closed subset of $(X, \mathcal{U})$.

The following proposition is a generalization of the well known result that regularity is preserved under continuous, open and closed surjections (Murdeshwar[4] Theorem 12.14, p. 206).

Lemma 2—If U and V are subsets of $(X, \mathcal{T})$, U is open and $U \subset V$, then cl $U \subset$ cl (int (cl V)).

PROOF : If U is open and $U \subset V$, then $U \subset$ cl V so that $U \subset$ int (cl V). Therefore cl $U \subset$ cl (int (cl (V)).

Proposition 8—If $f : (X, \mathscr{T}) \to (Y, \mathscr{U})$ is an open, continuous, α-closed surjection and $(X, \mathscr{T})$ is regular, then $(Y, \mathscr{U})$ is regular.

PROOF : Let $p \in Y$ and U be an open set in $(Y, \mathscr{U})$ containing p. Let $x \in X$ such that $f(x) = p$. Since $(X, \mathscr{T})$ is regular there is an open set V in $(X, \mathscr{T})$ such that $x \in V \subset \mathscr{T}\text{cl } V \subset f^{-1}(U)$ so that $p \in f(V) \subset f(\mathscr{T}\text{cl } V) \subset U$. Since f is α-closed, $f(\mathscr{T}\text{ cl } V)$ is α-closed and since f is open $f(V)$ is open so that, by Lemma 2, $\mathscr{U}\text{cl } f(V) \subset \mathscr{U}\text{ cl } (\mathscr{U}\text{ int } (\mathscr{U}\text{ cl } f(\mathscr{T}\text{cl } f(V))) \subset U$ and therefore $p \in f(V) \subset \mathscr{U}\text{cl } f(V) \subset U$.

It is well known that normality is preserved under closed, continuous surjections Murdeshwar[4] Theorem 15.3 (1). The following proposition is a generalization of this result.

Lemma 3—If U and V are subsets of $(X, \mathscr{T})$ and $U \cap V = \phi$ then $\text{int } (\text{cl } (\text{int } U)) \cap \text{int } (\text{cl } (\text{int } V) = \phi$.

PROOF : If $U \cap V = \phi$ then $\text{int } U \cap \text{int } V = \phi$, so that $\text{int } U \cap \text{cl } (\text{int } V) = \phi$. Therefore $\text{int } U \cap \text{int } (\text{cl } (\text{int } V)) = \phi$ which implies that $\text{cl } (\text{int } U) \cap \text{int } (\text{cl } (\text{int } V))$ ϕ, so that we have $\text{int } (\text{cl } (\text{int } U)) \cap \text{int } (\text{cl int } V)) = \phi$.

Proposition 9—If $f : (X, \mathscr{T}) \to (Y, \mathscr{U})$ is a continuous, α-closed surjection and $(X, \mathscr{T})$ is normal, then $(Y, \mathscr{U})$ is normal.

PROOF : Let A and B be closed sets of $(Y, \mathscr{U})$. Then there are open disjoint sets U and V in $(X, \mathscr{T})$ such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$, by normality of $(X, \mathscr{T})$. By Proposition 6 there are disjoint α-sets C and D in $(Y, \mathscr{U})$ such that $A \subset C$ and $B \subset D$ so that, by Lemma 3, $\text{int } (\text{cl } (\text{int } C))$ and $\text{int } (\text{cl } (\text{int } D))$ are disjoint open sets in $(Y, \mathscr{U})$ containing A and B respectively.

REFERENCES

1. D. Andrijević, *Mat. Vesnik* 36 (1984), 1–10.
2. D. S. Janković, and I. L. Reilly, *Indian J. pure appl. Math.* 16 (1985).
3. N. Levine, *Amer. Math. Mthly.* 70 (1963), 36–41.
4. M. G. Murdeshwar, *General Topology*. Halstead Press, New York, 1983.
5. A. S. Mashhour, I. A. Hasanein, and S. N. El-Deeb, *Acta Math.Hung.* 41 (1983), 213-18.
6. S. N. Maheshwari, and P. C. Jain, *J. Sci. Res.* 4 (1982), 67–70.
7. S. N. Maheshwari, and U. D. Tapi, *Madhya Bharati J. Univ. Saugar*, (1978-79) (to appear).
8. O. Njåstad, *Pac. J. Math.* 15 (1965), 961–70.
9. T. Noiri, *Atti. Acad. Nat. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* 54 (1973), 412-15.
10. I. L. Reilly and, M. K. Vamanamnrthy, *Acta Math. Hung.* 45 (1985), 27–32.

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PAIRWISE SET-CONNECTED MAPPINGS IN BITOPOLOGICAL SPACES

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In this paper, attempts have been made to generalize the concept of set-connected mappings in bitopological spaces. Such a mapping has been characterized and some of its properties have been studied. Finally, its relationships with pairwise continuous and pairwise weakly continuous mappings have been investigated.

1. Introduction

Kwak[5] introduced in a topological space a new type of mapping, called set-connected mapping which is weaker than a continuous map. Noiri[6] further investigated different properties of this mapping and showed that every weakly continuous surjection is set connected, but not conversely. In this note, we have studied the corresponding results in the bitopological setting. Since a bitopological space, as introduced by Kelly[4] as a natural structure, is a richer structure than that of a topological space, it is of much use and importance to study the generalizations of topological notions and implications in bitopological situation. Apart from this, it is interesting to notice the interactions of the two topologies in a bitopological space with reference to a study of a bitopological notion. These are the underlying reasons for extending and studying set connected maps in a bitopological space.

Section 2 opens up with the study of different properties of p-connected mapping (henceforth, the word "pairwise" shall be abbreviated as "$p-$"), specially in relation to its inherent applications to p-connected spaces, as introduced by Pervin[7]. Relations of this mapping with p-continuous and p-weakly continuous mappings have been discussed in section 3 and 4 respectively.

Throughout this note X and Y always represent bitopological spaces (X, Q_1, Q_2) and (Y, P_1, P_2) respectively and $f : X \to Y$ denotes a mapping from X into Y. If $A \subset X$, then the subspace A is the bitopological space $(A, (Q_1)_A, (Q_2)_A)$ induced by

(X, Q_1, Q_2). $Q_i-\text{cl}\, A$ and Q_i-int A respectively shall denote the closure and interior of A in (X, Q_i), for $i = 1, 2$. An ij-coset in any space (X, Q_1, Q_2) is a Q_i-closed and Q_j-open set in X and i, j can assume values 1 and 2 with $i \neq j$.

2. P-SET CONNECTED MAPPINGS

Definition 2.1—A bitopological space X is said to be 12-connected (21-connected) between A and B (where A, B are non-empty subsets of X) iff there exists no 12-coset (21-coset) F such that $A \subset F \subset X - B$.

Clearly, X is 12-connected between A and B iff it is 21-connected between B and A.

Definition 2.2—X is said to be p-connected between A and B (where A, B are nonempty subsets of X) iff X is 12-connected as well as 21-connected between A and B.

Remark 2.3 : If X is ij-connected between A and B and $C \supset A$, $D \supset B$, then X is ij-connected between C and D.

Definition 2.4—A mapping $f: X \to Y$ is said to be p-set connected iff $f(X)$ is ij-connected between $f(A)$ and $f(B)$ in the subspace bitopology whenever X is ij-connected between A and B, where $i, j = 1, 2$ and $i \neq j$.

Remark 2.5 : It is clear that the definition 2.4 can equivalently be modified by writting 12 or 21 in place of ij.

Lemma 2.6—If a subspace M of a bitopological space X is ij-connected between A and B then so is the whole space (where, as usual, $i, j = 1, 2$ and $i \neq j$).

PROOF : If there is any ij-coset F in the whole space X with $A \subset F \subset X-B$, then $F \cap M$ is ij-coset in M with $A \subset F \cap M \subset M - B$.

Lemma 2.7—A subspace M which is 12- (or 21-) coset in X, is ij-connected between A and B if X is ij-connected between two subsets A and B of M.

PROOF : If F is any 12-coset in M, then F is also 12-coset in X, as M is 12-coset in X. This fact implies that M is 12-connected between A and B if X is so. Now if X is 21-connected between A and B then it is 12-connected between B and A; so M is 12-connected between B and A which implies that M is 21-connected between A and B. Similar argument is applicable when M is 21-coset in X. Hence the lemma is proved.

Theorem 2.8—A mapping $f: X \to Y$ is p-set connected iff $f^{-1}(F)$ is 12-coset in X for any 12-coset F in $f(X)$.

PROOF : Let f be p-set connected and F be any 12-coset in $f(X)$. If $f^{-1}(F)$ is not 12-coset in X then X is 12-connected between $f^{-1}(F)$ and $X - f^{-1}(F)$ and therefore,

as f is p-set connected, $f(X)$ is 12-connected between $f(f^{-1}(F))$ and $f(X - f^{-1}(F))$. But $f(f^{-1}(F)) = F \cap f(X) = F$ and $f(X - f^{-1}(F)) = f(X) - F$ imply that F is not 12-coset in $f(X)$ — a contradiction. Hence $f^{-1}(F)$ is 12-coset in X.

Conversely, let the condition hold for f and let X be 12-connected between A and B. If $f(X)$ is not 12-connected between $f(A)$ and $f(B)$, then there exists a 12-coset F in $f(X)$ such that $f(A) \subset F \subset f(X) - f(B)$. But we find that $A \subset f^{-1}f(A) \subset f^{-1}(F)$, $f^{-1}(F) \subset X - B$ and $f^{-1}(F)$ is 21-coset in X. This contradicts that X is 12-connected between $f(A)$ and $f(B)$ i.e., f is p-set connected.

Remark 2.9 : (a) Similarly as in the above theorem, we can show that f is p-set connected iff $f^{-1}(F)$ is 21-coset in X for any 21-coset F in $f(X)$. (b) If f is p-set connected, then $f^{-1}(F)$ is 12-coset (21-coset) in X for any 12-coset (21-coset) F in Y.

Definition 2.10[7]—A bitopological space (X, Q_1, Q_2) is called pairwise connected (p-connected, according to our abbreviation) iff X cannot be expressed as the union of two non-empty disjoint sets A and B such that $(A \cap Q_1\text{–cl } B) \cup (Q_2\text{–cl } A \cap B) = \phi$.

It is shown[7] that $(X, Q_1\ Q_2)$ is pairwise connected iff X cannot be expressed as the union of two non-empty disjoint sets A and B such that A is Q_1-open and B is Q_2-open.

Now Lemmas 2.11 and 2.12, stated below, follow immediately.

Lemma 2.11—Every mapping $f: X \to Y$ such that $f(X)$ is p-connected, is p-set connected.

Lemma 2.12—Let $f: X \to Y$ be a p-set connected mapping. If X is p-connected, then $f(X)$ is p-connected.

Lemma 2.13—Let $f: X \to Y$ be p-set connected and $A\ (\subset X)$ be such that $f(A)$ is 12-coset (or 21-coset) in $f(X)$. Then the restriction $f|A : A \to Y$ is p-set connected.

Proof : Let A be 12-connected between C and D. By Lemma 2.6, X is 12-connected between C and D, so $f(X)$ is 12-connected between $f(C)$ and $f(D)$. As $f(A)$ is 12-coset (or 21-coset) in $f(X)$, by Lemma 2.7, $f(A)$ is 12-connected between $f(C)$ and $f(D)$. Hence by Remark 2.5, the lemma is proved.

Theorem 2.14—Let $f: X \to Y$ be p-set connected, b_i-open (i.e., $f: (X, Q_i) \to (Y, P_i)$ is open for $i = 1, 2$), surjection and $f^{-1}(y)$ be p-connected for each y of Y. Then for any 12- (or 21-) coset F in Y, F is p-connected iff $f^{-1}(F)$ is p-connected.

Proof : We prove the theorem for the case when F is 12-coset in Y. Similar will be the proof when F is 21-coset. Let $f^{-1}(F)$ be not p-connected for some 12-coset F in Y. Then there exists a 12-coset K with $\phi \neq K \neq f^{-1}(F)$, in the subspace bitopological space $(f^{-1}(F), Q_1/f^{-1}(F), Q_2/f^{-1}(F))$. We show that $f(K)$ is 12-coset in F with $\phi \neq f(K) \neq F$. As $f^{-1}(y)$ is p-connected, either $f^{-1}(y) \subset K$, or $f^{-1}(y) \subset f^{-1}(F) - K$, for all $y \in F$ and so $f(K) \neq F\ (\neq \phi)$ and $f(K) \cap f(f^{-1}(F) - K)$

$= \phi$. As f is surjective, $f(K) \cup f(f^{-1}(F) - K) = F$ and so $f(f^{-1}(F) - K) = F - f(K)$. Now f being *bi*-open $f/f^{-1}(F)$ is *bi*-open onto F and hence $f(K)$ is 12-coset in F. This implies that F is not p-connected. So F is p-connected implies $f^{-1}(F)$ is p-connected.

Next, since $f(f^{-1}(F)) = F$ which is 12-coset in Y, by Lemma 2.13, the restriction $f/f^{-1}(F) : f^{-1}(F) \to Y$ is p-set connected. Now by Lemma 2.12 it follows that $f/f^{-1}(F)(f^{-1}(F)) = F$ is p-connected if $f^{-1}(F)$ is p-connected. Hence the theorem.

Theorem 2.15—Let $f : X \to Y$ be a p-set connected mapping. Then for each $p \in f(X), f(Q_i - \text{cl}\, f^{-1}(p)) \subset C_p$, where C_p is any ij-coset in $f(X)$ (or in Y) containing the point p (where, as usual, $i, j = 1, 2$ and $i \neq j$).

PROOF : As C_p is ij-coset in $f(X)$ (or in Y) and f is p-set connected $f^{-1}(C_p)$ is ij-coset in X (by Theorem 2.8 and Remark 2.9 (b)). Also $f^{-1}(p) \subset f^{-1}(C_p)$, therefore $Q_i - \text{cl}\, (f^{-1}(p)) \subset f^{-1}(C_p)$. Hence $f(Q_i - \text{cl}\, (f^{-1}(p)) \subset ff^{-1}(C_p) \subset C_p$ and the theorem is proved.

Theorem 2.16—Let (X, Q_1, Q_2) be a bitopological space. If a sequence $\{p_n\}$ converges to p in (X, Q_i) and also if p has a p-connected Q_i-open neighbourhood, then there exists some n_0 such that X is p-connected between p and p_n, for all $n \geqslant n_0$ $(i = 1, 2)$.

PROOF : If p and p_n are contained in a p-connected set then X is p-connected between p and p_n.

Definition 2.17[1]—Let (X, Q_1, Q_2) be a bitopological space. Q_1 is said to be locally connected with respect to Q_2 if for each point x in X and each Q_1-open neighbourhood U of x there is a pairwise connected Q_1-open set G such that $x \in G \subset U$. (X, Q_1, Q_2) is pairwise locally connected if Q_1 is locally connected w. r. t. Q_2 and Q_2 is locally connected w.r.t. Q_1.

Corollary 2.18—If a sequence $\{x_n\}$ converges to x in (X, Q_i) and Q_i is locally connected w.r.t. Q_j then there exists some n_0 such that X is p-connected between p and p_n, for all $n \geqslant n_0$, where $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$.

Definition 2.19[3]—In a bitopological space (X, Q_1, Q_2), Q_i is said to be extremally disconnected w.r.t. Q_j iff Q_j-closure of any Q_i-open set is Q_i-open $(i, j = 1, 2$ and $i \neq j)$. The space is p-extremally disconnected iff Q_1 is extremally disconnected w.r.t. Q_2 and Q_2 is extremally disconnected q.r.t. Q_1.

Definition 2.20[4]—A bitopological space (X, Q_1, Q_2) is said to be pairwise Hausdorff if given distinct points x, y of X, there is a Q_1-open set U and a Q_2-open set V, such that $x \in U, y \in V, U \cap V = \phi$.

Theorem 2.21—Let $f : X \to Y$ be p-set connected and Y be p-Hausdorff and P_i is extremally disconnected w.r.t. P_j in Y. Then the graph $G(f)$ of f is closed in $(X \times Y, Q_j \times P_i)$ $(i, j = 1, 2$ and $i \neq j)$.

PROOF : Let $(x, y) \notin G(f)$ so that $y \neq f(x)$. Then y has a P_i-open neighbourhood whose P_j-closure, say V, does not contain $f(x)$ (since Y is p-Hausdorff). As P_i is extremally disconnected w.r.t. P_j, V is ji-coset in Y and so $f^{-1}(V)$ is ji-coset in X [by Remark 2.9 (b)] and $x \notin f^{-1}(V)$. Let $U = X - f^{-1}(V)$. Then U is Q_j-open in X containing x and $f(U) \cap V = \phi$. Thus $U \times V$ is an open neighbourhood of (x, y) in $(X \times Y, Q_j \times P_i)$ with $(U \times V) \cap G(f) = \phi$.
Hence $G(f)$ is closed in $(X \times Y, Q_j \times P_i)$.

Remark 2.22 : In the above theorem, one might expect that $G(f)$ is closed in $(X \times Y, Q_i \times P_i)$ (for $i = 1, 2$). That this is false is shown by the next example.

Example 2.23—Let $X = Y = R$ = real line. Let $Q_1 = P_1$ = cofinite topology on R and $Q_2 = P_2$ = discrete topology. Let $f : X \to Y$ be the identity map. Then f is clearly p-set connected, $(Y, P_1 P_2)$ is p-Hausdorff and P_1 is extremally disconnected w.r.t. P_2. But $G(f)$ is not closed in $(X \times Y, Q_1 \times P_1)$. In fact, $G(f)$ is dense in $(X \times Y, Q_1 \times P_1)$.

Theorem 2.24—Let $f : X \to Y$ be a mapping and $g : X \to (X \times Y, Q_1 \times P_1, Q_2 \times P_2)$ be given by $g(x) = (x, f(x))$, for $x \in X$. Then f is p-set connected if g is so.

PROOF : Let F be 12-coset in $f(X) \subset Y$. Then $X \times F$ is 12-coset in the subspace $X \times f(X) \subset X \times Y$ (we note that the topologies on $X \times f(X)$, namely $Q_i \times P_i / f(X)$ and that inherited as a subspace from $Q_i \times P_i$ are identical, for $i = 1, 2$). As $g(X) \subset X \times f(X)$, $(X \times F) \cap g(X)$ is 12-coset in $g(X)$ and so $g^{-1}[(X \times F) \cap g(X)]$ is 12-coset in X by Theorem 2.8. But $g^{-1}[(X \times F) \cap g(X)] = g^{-1}(X \times F) = f^{-1}(F)$. Hence f is p-set connected.

Remark 2.25 : Converse of the above theorem is false as is seen from example below.

Example 2.26—Let $X = Y = R$, the real line; $Q_1 = P_1$ = the left hand topology and $Q_2 = P_2$ = the right hand topology. Let $f : X \to Y$ be defined as $f(x) = x$ if $x \neq 0$ and $f(0) = -1$. Here f is p-set connected as Y has no nontrivial 12- or 21-coset. Now $[0, \infty) \times [0, \infty)$ is closed in $Q_1 \times P_1$, $(0, \infty) \times (0, \infty)$ is open in $Q_2 \times P_2$ and $([0,\infty)) \times [0,\infty)) \cap g(X) = ((0,\infty) \times (0,\infty)) \cap g(X) = K$ (say). Thus K is 12-coset in the subspace $g(X)$ of $X \times Y$. But X has no non-trivial 12-coset whereas $g^{-1}(K) = (0, \infty)$. Hence g is not p-set connected.

3. P-SET CONNECTED MAPPINGS AND P-CONTINUOUS MAPPINGS

Definition 3.1[7]—A mapping $f : X \to Y$ will be said to be p-continuous iff the induced maps $f : (X, Q_1) \to (Y P_1)$ and $f : (X, Q_2) \to (Y, P_2)$ are continuous.

By virtue of Theorem 2.8 it follows that :

Theorem 3.2—Every p-continuous mapping is p-set connected. The following example shows that the converse is false.

Example 3.3—Let $X = Y = R$, the real line, and $P_1 = Q_1$ = the usual topology on R and $P_2 = Q_2$ = the countable complement topology on R. Let $f : X \to Y$ be defined by $f(x) = x$, $x \neq 0$ and $f(0) = 1$. Then f is p-set connected, since there is no proper 12- or 21-coset in $f(X) = Y - \{0\}$. But $f : (X, Q_1) \to (Y, P_1)$ is not continuous at $x = 0$ and hence $f : X \to Y$ is not p-continuous.

Now, we proceed to find the conditions under which a p-set connected mapping is p-continuous.

Theorem 3.4—Let $f : X \to Y$ be p-set connected. If Y be p-Hausdorff and P_1 (or P_2) is extremally disconnected w.r.t. P_2 (or P_1), then $f/C : C \to Y$ is constant for every p-connected subset C of X.

PROOF : Let $x, y \in C$ and $x \neq y$. If possible, let $f(x) \neq f(y)$ in Y. Then there exists 21-coset V in Y such that $f(x) \in V$ and $f(y) \notin V$, since Y is p-Hausdorff and P_1 is extremaly disconnected w.r.t. P_2. Now, $f^{-1}(V)$ is 21-coset in X as f is p-set connected and therefore, $f^{-1}(V) \cap C$ is a non-empty proper 21-coset in the subspace C. This condradicts that C is p-connected. Hence $f(x) = f(y)$, for all $x, y \in C$, i.e., $f/C : C \to Y$ is constant.

Corollary 3.5—Let $f : X \to Y$ be p-set connected, Y be p-Hausdorff and P_1 (or P_2) is extremally disconnected w.r.t. P_2 (or P_1) and for each $x \in X$ it has a Q_i-open p-connected neighbourhood. Then $f^{-1}(y)$ is Q_i-open in X for each $y \in Y$, for $i = 1, 2$.

PROOF : Follows from the fact that for each $x \in f^{-1}(y)$ there is a Q_i-open p-connected set U with $x \in U \subset f^{1}(y)$.

Remark 3.6 : In the above corollary, if for each $x \in X$, x has a Q_1-open p-connected neighbourhood and a Q_2-open p-connected neighbourhood, then f is p-continuous Thus

Corollary 3.7—If $f : X \to Y$ be p-set connected, X be p-locally connected and Y be p-extremally disconnected and p-Hausdorff, then f is p-continuous.

Definition 3.8—A space (X, Q_1, Q_2) is p-c-compact iff every cover of any Q_i-closed set in X by Q_j-open sets of X has finite subfamily whose union is Q_i-dense in the Q_i-closed set, for $i, j = 1, 2$ and $i \neq j$.

Theorem 3.9—Let Y be p-extremally disconnected, p-c-compact and p-Hausdorff. Then $f : X \to Y$ is p-continuous if it is p-set connected.

PROOF : Let f be not p-continuous. Then there exists a P_i-closed set F in Y such that $f^{-1}(F)$ is not Q_i-closed in X (for $i = 1$ or 2). Let $x \in Q_i - \text{cl}(f^{-1}(F)) - f^{-1}(F)$. Then X is ij-connected between $f^{-1}(F)$ and x. So $f(X)$ is ij-connected between $f(f^{-1}(F))$ and $f(x)$. By Lemma 2.6 and Remark 2.3, we obtain that Y is ij-connected between $F (\supset ff^{-1}(F))$ and $f(x)$. Now as Y is p-Hausdorff, for each

$y \in F$ there exists a P_j-open neighbourhood V_y of y in Y such such that $f(x) \notin P_i-$cl (V_y). Then the family $\{V_y : y \in F\}$ is a cover of F by P_j-open sets in Y. By p-c-compactness, there exists finite set of points $y_1, y_2, \ldots, y_n$ in F such that $F \subset P_i-$cl $(\bigcup_{r=1}^{n} V_{y_r}) = V$ (say). Then V is ij-coset in Y, since Y is p-extremally disconnected and P_i-cl $(\bigcup_{r=1}^{n} V_{y_r}) = \bigcup_{r=1}^{n} P_i-$cl (V_{y_r}). Also, $f(x) \notin V$, since $f(x) \notin P_i-$cl (V_y), for any $y \in F$. This contradicts that Y is ij-connected between F and $f(x)$. Hence f is p-continuous.

4. P-SET CONNECTED MAPPING AND P-WEAKLY CONTINUOUS

Definition 4.1[2]—A mapping $f : (X, Q_1, Q_2) \to (Y, P_1\ P_2)$ is said to be $Q_1\ P_1$-weakly continuous w.r.t. P_2 if for each point x of X and each P_1-neighbourhood V of $f(x)$ there exists a Q_1-neighbourhood U of x such that $f(U) \subset P_2-$cl (V).

The mapping f will be called p-weakly continuous if it is $Q_1\ P_1$ weakly continuous w.r.t. P_2 as well as $Q_2\ P_2$ weakly continuous w.r.t. P_1

Lemma 4.2[2]—A mapping $f : X \to Y$ is $Q_1\ P_1$-weakly continuous w.r.t. P_2 iff for each P_1-open set V of Y, $f^{-1}(V) \subset Q_1-$int $(f^{-1}(P_2-$cl $V))$.

Lemma 4.3[2]—If a mapping $f : X \to Y$ is $Q_1\ P_1$ weakly continuous w.r.t. P_2 then Q_1-cl $(f^{-1}(V)) \subset f^{-1}(P_1-$cl $V)$, for any P_2-open set V.

Theorem 4.4—If a surjection $f : (X, Q_1, Q_2) \to (Y, P_1, P_2)$ is p-weakly continuous, then f is p-set connected.

PROOF: Let V be 12-coset in $Y (= f(X))$. Since V is P_2-open, by Lemma 4.2 $f^{-1}(V) \subset Q_2-$int $(f^{-1}(P_1-$cl $V)) = Q_2-$int $(f^{-1}(V))$, since V is P_1-closed. This implies $f^{-1}(V) = Q_2-$int $(f^{-1}(V))$ and hence $f^{-1}(V)$ is Q_2-open in X. Again by Lemma 4.3, Q_1-cl $(f^{-1}(V)) \subset f^{-1}(P_1-$cl $V) = f^{-1}(V)$ so that Q_1-cl $(f^{-1}(V)) = f^{-1}(V)$ and hence $f^{-1}(V)$ is Q_1-closed in X. Thus $f^{-1}(V)$ is 12-coset in X. Then by Theorem 2.8, f is p-set connected.

Remark 4.5 : The following example shows that in the above theorem the condition "p-weakly continuous" can not be relaxed by $Q_1\ P_1$-weakly continuous w.r.t. P_2 or $Q_2\ P_2$-weakly continuous w.r.t. P_1.

Example 4.6—Let $X = Y = R$, the real line, Q_1, Q_2 be respectively the discrete and indiscrete topology on R; P_1 the lower limit topology and P_2 the usual topology on Y. Suppose $f : (X, Q_1, Q_2) \to (Y, P_1, P_2)$ be the identity mapping. Now $A = [0, \infty)$ is 21-coset in Y. But $f^{-1}(A) = A$ is not 21-coset in X. Therefore, f is not p-set connected. But it is clear that f is $Q_1\ P_1$ weakly continuous w.r.t. P_2 and is not p-weakly continuous.

Remark 4.7 : The converse of the Theorem 4.4 is false as is seen from the example below.

Example 4.8—Let $X = Y = R$; Q_1, Q_2 respectively denote the cofinite and discrete topology on X and let $P_1 = P_2 =$ usual topology on Y. Let $f : (X, Q_1, Q_2) \to (Y, P_1, P_2)$ be the identity map. Since (Y, P_1, P_2) is p-connected, f is p-set connected. Now consider $o \in X$ and $V = (-1, 1)$ is a P_1-neighbourhood of $f(o) = 0$ in Y. Then there cannot exist any Q_1-open neighbourhood U of o such that $f(U) \subset P_2-\mathrm{cl}V = [-1, 1]$. Hence f is not p-weakly continuous.

Theorem 4.9—Let $f : X \to Y$ be p-set connected and Y is p-extremally disconnected, then f is p-weakly continuous.

Proof : Let $x \in X$ and $f(x) \in V \in P_i$ $(i = 1, 2)$. Then $P_j-\mathrm{cl}V$ is ji-coset in Y and so $(P_j-\mathrm{cl}\ V) \cap f(X)$ is ji-coset in $f(X)$ $(i, j = 1, 2; i \neq j)$. Since f is p-set connected $f^{-1}(P_j-\mathrm{cl}\ V \cap f(X)) = U$ (say) is ji-coset in X. Then U is a Q_i-open neighbourhood of x such that $f(U) \subset P_j-\mathrm{cl}\ V$ $(i, j = 1, 2$ and $i \neq j)$, Hence f is p-weakly continuous. We shall sharpen the result in Theorem 4.9. For that we require the following definition.

Definition 4.10[2]—A mapping $f : (X, Q_1, Q_2) \to (Y, P_1, P_2)$ is called Q_1 P_1-almost continuous w.r.t. P_2 iff for each x of X and each P_1-neighbourhood V of $f(x)$ in Y there exists a Q_1-neighbourhood U of x in X such that $f(U) \subset P_1-\mathrm{int}\ (P_2-\mathrm{cl}\ V)$. If f is, in addition, Q_2 P_2-almost continuous w.r.t. P_1, then f is called p-almost continuous.

Remark 4.11 : Since every p-almost continuous map is obviously p-weakly continuous, from Theorem 4.4 it follows that a p-almost continuous surjection is p-set connected.

Now we observe that p-weakly continuity in Theorem 4.9 can be replaced by p-almost continuity.

Theorem 4.12—Let Y be p-extremally disconnected and $f : X \to Y$ be p-set connected. Then f is p-almost continuous.

Proof : The same proof of Theorem 4.9 can be carried over here, noticing that $P_j-\mathrm{cl}\ V = P_i-\mathrm{int}\ (P_j-\mathrm{cl}\ V)$ $i, j = 1, 2$ and $i \neq j)$, since Y is p-extremally disconnected.

References

1. T. Bîrsan, *An. Sti. Univ. Al. T Cuza Iasi. See Ia Math.* N. S. **14** (1968), 293-96.
2. S. Bose, and D. Sinha, *Bull. Calcutta Math. Soc.* **74** (1982), 195-206.
3. M. C. Datta, *J. Austr. Math. Soc.* **14** (1972), 119-28.
4. J. C. Kelly, *Proc. London Math. Soc.* (3) **13** (1963), 71-89.
5. Jin Ho Kwak, *Kyungpook Math. J.* **11** (1971), 169-72.
6. T. Noiri, *Kyungpook Math. J.* **16** (1976), 243-46.
7. W. J. Pervin, *Indag. Math.* **29** (1967), 369-72.

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ON α-STARLIKE AND α-CLOSE-TO-CONVEX FUNCTIONS WITH RESPECT TO n-SYMMETRIC POINTS

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We define the class $K_n(\alpha; h)$—the class of α-starlike functions with respect to n symmetric points– consisting of $f(z) = z + a_2z^2 + \cdots$ satisfying $\frac{f(z)\, f'(z)}{z} \neq 0$ in E and $\frac{\alpha z\,(zf'(z))' + (1-\alpha)\, z f'(z)}{\alpha z f'_n(z) + (1-\alpha) f_n(z)} \prec h(z)$, $\alpha \geqslant 0$,

where $h(z)$ is a given convex univalent function in E with $h(0) = 1$ Re $h(z) > 0$ in E and $f_n(z) = \frac{1}{n}\sum_{j=0}^{n-1} \omega^{-j} f(\omega^j z)$ with $\omega = \exp(2\pi i/n)$. In this paper we prove theorems which establish that $K_n(\alpha; h) \subset K_n(0; h)$, $K_n(\alpha; h)$ is closed under an integral operator and the coefficient estimate for this class. Also we define two class $C_n(\alpha; h)$ of functions and investigate the properties of this class.

INTRODUCTION

Let $E = \{z : |z| < 1\}$ be the open unit disc in $\mathbb{C}$, $H(E)$ denote the class of functions $f(z)$ holomorphic in E. Let $A = \{f \in H(E) : f(0) = f'(0) - 1 = 0\}$. For a given positive integer n let $\omega = \exp(2\pi i/n)$ and

$$f_n(z) = \frac{1}{n}\sum_{j=0}^{n-1} \omega^{-j} f(\omega^j z),\ z \in E. \qquad \ldots(1)$$

Here we introduce new subclasses of S-the class of normalized univalent functions and study certain properties of these classes. First let us define the class $K_n(\alpha; h)$ which unifies the classes of convex univalent functions with respect to symmetric points and starlike univalent functions with respect to symmetric points.

Definition 1—The function $f(z) \in A$ and $\frac{f(z)\, f'(z)}{z} \neq 0$ in E is said to be α-starlike with respect to n-symmetric points if it satisfies

$$\frac{\alpha z\,(zf'\,z))' + (1-\alpha)\,zf'(z)}{\alpha z\, f_n'(z) + (1-\alpha) f_n(z)} \prec h(z), \quad \text{for } \alpha \geqslant 0$$

where $h(z)$ is a given convex univalent function in E with $h(0) = 1$, $\operatorname{Re} h(z) > 0$ and $\prec$ means subordination. Let us denote the class of such functions as $K_n(\alpha;h)$.

When $n = 1$ and $h(z) = \dfrac{1-z}{1+z}$ this reduces to the class $K(\alpha)$ of α-starlike functions studied by Pascu and Podaru[7]. Also $K_2\left(1, \dfrac{1-z}{1+z}\right)$ coincides with the class Cs-defined by Das and Singh[1], $K_2\left(0, \dfrac{1-z}{1+z}\right)$ is the class of starlike functions with respect to symmetric points investigated by Sakaguchi[8]. Of course $K_n(0;h)$ is the class of starlike functions with respect to n-symmetric points of Mocanu[3]. Also here we define another class $C_n(\alpha;h)$ of α-close-to-convex functions with respect to n-symmetric points.

Definition 2—A function $f(z) \in A$ such that $\dfrac{f(z)f'(z)}{z} \neq 0$ in E is said to be α-close-to-convex with respect to n-symmetric points if it satisfies

$$\frac{\alpha z\,(zf'(z))' + (1-\alpha)\,zf'(z)}{\alpha z\phi_n'(z) + (1-\alpha)\,\phi_n(z)} \prec h(z), \text{ for } \alpha \geqslant 0$$

where $\phi_n(z) = \dfrac{1}{n}\sum_{j=0}^{n-1} \omega^{-j}\,\phi(\omega^j z)$ with $\phi(z) \in K_n(\alpha;h)$ and $h(z)$ is as defined in Definition 1.

This class generalizes the classes defined by Pascu[6] and Das and Singh[1]. Let us investigate these two classes in this paper. In establishing our main results regarding these classes we often make use of the following lemma due to Padmanabhan and Parvatham[4].

Lemma A—Let $\beta, \gamma \in \mathbb{C}$, $h \in H(E)$ be convex univalent in E with $h(0) = 1$ and $\operatorname{Re}(\beta h(z) + \gamma) > 0$, $z \in E$ and let $q \in H(E)$ with $q(0) = 1$ and $q(z) \prec h(z)$, $z \in E$. If $p(z) = 1 + p_1 z + \dots$ in analytic in E, then

$$p(z) + \frac{zp'(z)}{\beta\, q(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Now, we prove a lemma which we use in the sequel.

Lemma—Let $f(z) \in K_n(\alpha;h)$. Then $f_n(z)$ defined by (1) is in $K_1(\alpha;h)$. Further

$$\frac{zf_n'(z)}{f_n(z)} \prec h(z) \text{ in } E.$$

PROOF : First let us consider

$$\frac{\alpha z\,(z f_n'(z))' + (1-\alpha)\, z f_n'(z)}{\alpha z f_n'(z) + (1-\alpha) f_n(z)} = \frac{\frac{1}{n}\sum_{j=0}^{n-1} \alpha z\,(zf(\omega^j z))' + (1-\alpha)\, zf'(\omega^j z)}{\alpha z f_n(z) + (1-\alpha) f_n(z)}$$

$$= \frac{1}{n}\sum_{j=0}^{n-1} \frac{\alpha z\,(zf'(\omega^j z))' + (1-\alpha\, zf'(\omega^j z)}{\alpha z f_n'(z) + (1-\alpha\, f_n(z)}.$$

Since $f(z) \in K_n(\alpha; h)$ each term under the summation on the right side of the above equality is subordinate to $h(z)$. Hence there exist ζ_j' s in E such that

$$\frac{\alpha z\left(zf_n'(z)\right)' + (1-\alpha)\, zf_n'(z)}{\alpha zf_n'(z) + (1-\alpha) f_n(z)} = \frac{1}{n}\sum_{j=0}^{n-1} h(\zeta_j) = h(\zeta_0)$$

$\zeta_0 \in E$ since $h(E)$ is convex. Thus $f_n(z) \in K_1(\alpha, h)$.

Since,

$$\frac{\alpha z\left(zf_n'(z)\right)' + (1-\alpha)\, zf_n'(z)}{\alpha zf_n'(z) + (1-\alpha) f_n(z)} = p(z) + \frac{\alpha z p'(z)}{(1-\alpha) + \alpha p(z)}$$

where $p(z) = \dfrac{zf_n'(z)}{f_n(z)}$, an application of a result by Eenigenburg *et al.*[2] now yields

$$\frac{zf_n'(z)}{f_n(z)} = p(z) \prec h(z).$$

Theorem 1—We have the following inclusion relation :

$$K_n(\alpha; h) \subset K_n(0; h).$$

PROOF : Let $f(z) \in K_n(\alpha; h)$, $p(z) = \dfrac{zf'(z)}{f_n(z)}$ and $q(z) = \dfrac{zf_n'(z)}{f_n(z)}$.

Then

$$\frac{\alpha z\,(zf'(z))' + (1-\alpha)\, zf'(z)}{\alpha zf_n'(z) + (1-\alpha) f_n(z)}$$

(equation continued on p. 1117)

$$= \frac{\alpha z\,[f'_n(z)\,p(z) + p'(z)\,f_n(z)] + (1-\alpha)\,[p(z)\,f_n(z)]}{\alpha z f'_n(z) + (1-\alpha)\,f_n(z)}$$

$$= \frac{\alpha z p'(z) + \dfrac{\alpha z f'_n(z)}{f_n(z)}\,p(z) + (1-\alpha)\,p(z)}{\alpha z \dfrac{f'_n(z)}{f_n(z)} + (1-\alpha)}$$

$$= \frac{\alpha z p'(z) + p(z)\left[(1-\alpha) + \dfrac{\alpha z f'_n(z)}{f_n(z)}\right]}{(1-\alpha) + \dfrac{\alpha z f'_n(z)}{f_n(z)}}$$

$$= \frac{\alpha z p'(z)}{(1-\alpha) + \alpha q(z)} + p(z) \prec h(z)$$

since $f(z) \in K_n(\alpha; h)$. Here $q(z) \prec h(z)$ by the above lemma. Now an application of lemma A gives $p(z) \prec h(z)$ in E which completes the proof of the theorem.

Remark 1 : When $n = 1$ and $h(z) = \dfrac{1-z}{1+z}$ we get a result of Pascu[5] as a particular case of our result.

Now, we establish a theorem which shows that this class is closed under an integral operator.

Theorem 2—$F(z) = \dfrac{1}{\alpha z^{1/\alpha - 1}} \displaystyle\int_0^z t^{(1/\alpha)-2} f(t)\,dt \in K_n(\alpha; h)$

whenever $f(z) \in K_n(\alpha; h)$ for $\alpha > 0$.

Proof : Consider $F_n(z)$ defined by (1) with $F(z)$ in the place of $f(z)$. That is, $F_n(z) = \dfrac{1}{n} \displaystyle\sum_{j=0}^{n-1} \omega^{-j} F(\omega^j z)$. It is easy to see that $F_n(z) = \dfrac{1}{\alpha z^{(1/\alpha)-1}} \displaystyle\int_0^z t^{(1/\alpha)-2} f_n(t)\,dt$.

Differentiating this with respect to z and simplifying we get

$$\alpha z F'_n(z) + (1-\alpha)\,F_n(z) = f_n(z). \qquad \ldots(2)$$

From the definition of $F(z)$ we get, $\alpha zF'(z) + (1-\alpha) F(z) = f(z)$ which on differentiation gives

$$\alpha z (zF'(z))' + (1-\alpha) zF'(z) = zf'(z). \qquad \ldots(3)$$

From (2) and (3) we get

$$\frac{\alpha z (zF'(z))' + (1-\alpha) zF'(z)}{\alpha zF'_n(z) + (1-\alpha) F_n(z)} = \frac{zf'(z)}{f_n(z)} \prec h(z)$$

because of Theorem 1 and the fact that $f(z) \in K_n(\alpha;h)$. This implies that $F(z) \in K_n(\alpha;h)$.

We now obtain the coefficient estimates for the class $K_n(\alpha;h)$.

Theorem 3—Let $f(z) = z + \sum_{i=2}^{\infty} a_i z^i \in K_n(\alpha;h)$. Then $\forall\, k \geqslant 1,\ n = 1, 2, \ldots$

$$|a_{kn+1}| \leqslant \frac{|h_1| \left(1+\frac{|h_1|}{n}\right)\left(1+\frac{|h_1|}{2n}\right)\cdots\left(1+\frac{|h_1|}{(k-1)n}\right)}{kn(kn\alpha+1)} \qquad \ldots(4)$$

for i satisfying $(k-1)n+1 < i < kn+1,\ \forall\, k \geqslant 1,\ n = 1, 2, \ldots$

$$|a_i| \leqslant \frac{|h_1| \left(1+\frac{|h_1|}{n}\right)\left(1+\frac{|h_1|}{2n}\right)\cdots\left(1+\frac{|h_1|}{(k-1)n}\right)}{i(i\alpha+1-\alpha)}$$

where $h(z) = 1 + h_1 z + \ldots.$

PROOF: Let $\dfrac{\alpha z (zf'(z))' + (1-\alpha) zf'(z)}{\alpha zf'_n(z) + (1-\alpha) f_n(z)} = p(z) = 1 + p_1 z + p_2 z^2 + \ldots$

where $f_n(z) = \frac{1}{n}\sum_{j=0}^{n-1} \omega^{-j} f(\omega^j z)$ and so $f'_n(z) = \frac{1}{n}\sum_{j=0}^{n-1} f'(\omega^j z)$.

Since $f(z) \in K_n(\alpha;h)$, $p(z) \prec h(z) = 1 + h_1 z + h_2 z^2 + \ldots.$ It is well known that $\forall\, i \geqslant 2$, $|p_i| \leqslant |h_1|$. Now,

$$\alpha z (zf'(z))' + (1-\alpha) zf'(z) = (1 + p_1 z + p_2 z^2 + \ldots)(\alpha zf'_n(z) + (1-\alpha) f_n(z))$$

(*equation continued on p.* 1119)

$$= (1 + p_1 z + \dots)\left(z + \sum_{i=2}^{\infty} a_i \left(\frac{\alpha(i-1)+1}{n}\right)\left(\sum_{j=0}^{n-1} \omega^{(i-1)j} z^i\right)\right).$$

Comparing the coefficients on either side,

$$|a_i|(i^2\alpha + i(1-\alpha)) \leqslant |h_1| + |h_1||a_2|\left(\frac{\alpha+1}{n}\right)\sum_{j=0}^{n-1}\omega^j$$

$$+ |h_1||a_3|\left(\frac{2\alpha+1}{n}\right)\sum_{j=0}^{n-1}\omega^{2j}$$

$$+ \dots + |h_1||a_{i-1}|\left(\frac{(i-2)\alpha+1}{n}\right)$$

$$\times \sum_{j=0}^{n-1}\omega^{(i-2)j}$$

$$+ |a_i|\left(\frac{(i-1)a+1}{n}\right)\sum_{j=0}^{n-1}\omega^{(i-1)j}. \quad \dots(5)$$

Also we make use of the fact that $\sum_{j=0}^{n-1} \omega^{(i-1)j} = n$ when $i = kn + 1$, $\forall\, k > 1$; $\sum_{j=0}^{n-1} \omega^{(i-1)j} = 0$ when $i \neq kn + 1$, $\forall\, k \geqslant 1$.

Hence let us consider the two cases separately.

Case (i) —First let us consider the case $i = kn + 1$. Then we establish (4) for all $k > 1$, by the method of induction, In this case (5) reduces to

$$kn(kn\alpha + 1)|a_{kn+1}| \leqslant |h_1| + |h_1|(n\alpha+1)|a_{n+1}|$$

$$+ |h_1|(2n\alpha+1)|a_{2n+1}| + \dots +$$

$$+ |h_1|((k-1)k\alpha+1)|a_{(k-1)n+1}|.$$

Let us assume that the estimate (4) is true for $k = 1, 2, \dots, (j-1)$. Hence the above inequality yields,

$$|jn(jn\alpha+1)|a_{jn+1}| \leqslant |h_1| + |h_1|\frac{|h_1|}{n} + \frac{|h_1||h_1|\left(1+\frac{|h_1|}{n}\right)}{2n}$$

$$+ \dots + \frac{|h_1||h_1|\left(1+\frac{|h_1|}{n}\right)\dots\left(1+\frac{|h_1|}{(j-2)n}\right)}{(j-1)n}$$

which on simplification gives

$$| a_{jn+1} | \leqslant \frac{| h_1 | \left(1+ \frac{| h_1 |}{n}\right) \left(1+ \frac{| h_1 |}{2n}\right) + \ldots + \left(1+ \frac{| h_1 |}{(j-1)n}\right)}{jn\,(jn\alpha + 1)}.$$

This shows that (4) is true for $k = j$ also. It is obvious from (5) that (4) is true for $k = 1$. Hence it is true for $\forall\, k \geqslant 1$.

Case (ii)—Let $i \neq jn + 1$ for $j \geqslant 1$. Then there exists *a* k such that $(k - 1)\, n+1 < i < kn + 1$. Here (5) reduces to

$$\begin{aligned} i\,(i\alpha + 1 - \alpha)\, | a_i | \leqslant | h_i | &+ | h_1 |\, (n\alpha + 1)\, | a_{n+1} | \\ &+ | h_1 |\, (2n\alpha+1)\, | a_{2n+1} | \\ &+ \ldots + | h_1 |\, ((j-1)\, n\alpha + 1)\, | a_{(j-1)n+1} | . \end{aligned}$$

Using (4) the above inequality becomes for $(k - 1)\, n + 1 < i < kn + 1$

$$\begin{aligned} i\,(i\alpha + 1-\alpha)\, | a_i | \leqslant | h_1 | + | h_1 | \frac{| h_1 |}{n} &+ \frac{| h_1 || h_1 | \left(1 + \frac{| h_1 |}{n}\right)}{2n} \\ &+ \ldots + | h_1 || h_1 | \left(1+ \frac{| h_1 |}{n}\right) \ldots \left(1+\frac{| h_1 |}{(k-2)n}\right) \end{aligned}$$

which on simplification gives

$$| a_i | \leqslant \frac{| h_1 | \left(1+ \frac{| h_1 |}{n}\right) \left(1+\frac{| h_1 |}{2n}\right) \ldots \left(1+ \frac{| h_1 |}{(k-1)n}\right)}{i\,(i\alpha + 1 - \alpha)} .$$

Remark 2: When h is a general convex univalent function we can not ascertain anything about the sharpness of the coefficient inequality. However when $n = 2$, $\alpha = 0$ and $h\,(E)$ is the half plane Re $z > 0$, we get $| a_i | \leqslant 1$, $\forall\, i \geqslant 2$, the sharp estimate due to Sakaguchi[8]. When $n = 2$, $\alpha = 1$ and $h\,(E)$ is the half plane Re $z > 0$ we have for $\forall\, k \geqslant 1$, the sharp estimate $| a_k | \leqslant \frac{1}{k}$, a result of Das and Singh[1].

Theorem 4—We have $C_n\,(\alpha;h) \subset C_n\,(0;h)$.

PROOF : Let $f\,(z) \in C_n\,(\alpha;h)$. Setting $p\,(z) = \frac{zf\,(z)}{\phi_n\,(z)}$ and $q\,(z) = z\,\frac{\phi'_n\,(z)}{\phi_n\,(z)}$ we have

$$\frac{\alpha z\,(zf'\,(z))' + (1-\alpha)\, zf'\,(z)}{\alpha z\phi'_n\,(z) + (1-\alpha)\,\phi_n\,(z)} = \frac{\alpha zp'\,(z) + p\,(z)\left(\alpha z\,\frac{\phi'_n\,(z)}{\phi_n\,(z)} + 1-\alpha\right)}{\alpha z\,\frac{\phi'_n\,(z)}{\phi_n\,(z)} + (1-\alpha)}$$

(*equation continued on p.* 1121)

$$= p(z) + \frac{\alpha z p'(z)}{\alpha q(z) + (1-\alpha)} \prec h(z)$$

since $f(z) \in C_n(\alpha;h)$. Here $q(z) \prec h(z)$ (by our lemma). Again an application of lemma A yields $p(z) = \dfrac{zf'(z)}{\phi_n(z)} \prec h(z)$ which establishes the theorem.

Now let us get an integral representation for functions belonging to the class $C_n(\alpha;h)$.

Theorem 5 – A function $f(z) \in C_n(\alpha;h)$ if and only if there exists a function $G(z)$ in $H(E)$ with $G(o)=0$ such that $\dfrac{zG'(z)}{G(z)} \prec h(z)$ and an analytic function $p(z)$ with $p(0) = 1$ and $p(z) \prec h(z)$ in E such that

$$f'(z) = \frac{1}{\alpha z^{1/\alpha}} \int_0^z p(t)\, G(t)\, t^{(1/\alpha)-2}\, dt, \text{ if } \alpha \neq 0$$

and

$$f'(z) = \frac{p(z)\, G(z)}{z}, \text{ if } \alpha = 0.$$

PROOF : Let $f(z) \in C_n(\alpha;h)$. Then there exists a

$$\phi \in K_n(\alpha;h) \text{ such that } \frac{\alpha z\,(zf'(z))' + (1-\alpha)\, zf'(z)}{\alpha z\, \phi_n'(z) + (1-\alpha)\, \phi_n(z)} \prec h(z)$$

where $\phi_n(z)$ is defined by (1). By our lemma $\phi_n \in K_1(\alpha;h)$; That is,

$$\frac{\alpha z\left(z\,\phi_n'(z)\right)' + (1-\alpha)\, z\phi_n'(z)}{\alpha z\, \phi_n'(z) + (1-\alpha)\, \phi_n(z)} \prec h(z).$$

Hence we can write

$$\frac{\alpha z\left(z\,\phi_n'(z)\right)' + (1-\alpha)\, z\,\phi_n'(z)}{\alpha z\, \phi_n'(z) + (1-\alpha)\, \phi_n(z)} = \frac{zG'(z)}{G(z)}$$

where $\dfrac{zG'(z)}{G(z)} \prec h(z)$ which on integration gives,

$$\alpha z\, \phi_n'(z) + (1-\alpha)\, \phi_n(z) = G(z)$$

and so,

$$\frac{\alpha z\,(zf'(z))' + (1-\alpha)\, zf'(z)}{G(z)} = p(z) \prec h(z)$$

that is,

$$\alpha z\,(zf'(z))' + (1-\alpha)\, zf'(z) = p(z)\, G(z).$$

When $\alpha \neq 0$ multiplying by $\frac{z^{(1/\alpha)-2}}{\alpha}$ and integrating we get

$$f'(z) = \frac{1}{\alpha z^{1/\alpha}} \int_0^z p(t)\, G(t)\, t^{(1/\alpha)-2}\, dt.$$

Conversely it is easily seen that if $f(z)$ has the above integral representation then $f(z) \in C_n(\alpha;h)$

References

1. R. N. Das, and P. Singh, *Indian J. pure appl. Math.* **8** (1977), 864–72.
2. P. Eeingenburg, S. S. Miller, P. T. Mocanu, and M. O. Reade, *Rev. Roumaine Math. Pures Appl* **29** (1984), 567–73.
3. P. T. Mocanu, *Bull. Math. de la soc. sci. Math de la R. S. de Roumanie* **28** (1984), 47–50.
4. K. S. Padmanabhan, and R. Parvatham, *Bull. Math. de la soc. Sci. Math. de la R. S. de Roumanie*. (To appear).
5. N. N. Pascu, *Bul,. Univ Brasov. Seria C*, **19** (1977).
6. N. N. Pascu, Lecture Notes in Mathematics, 743, Springer-Verlag, (1976), pp. 341–35.
7. N. N. Pascu, and V. Podaru, Lecture Notes in Mathematics, No. 1013, Springer-Verlag (1981), pp. 336–49.
8. K. Sakaguchi, *J. Math. Soc. Japan* **11** (1959), 72–80.

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INVERSE AND REVERSIBILITY OF QUASIMATRIX TRANSFORMATION

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The object of the paper is to determine conditions under which the quasi method associated with an infinite matrix admits an inverse transformation.

1. INTRODUCTION AND DEFINITION

Vermes[8] pointed out that there is a close relationship between the summability properties of a matrix $A = (a_{nk})$ regarded as a sequence-to-sequence transformation and those of its transpose $A^* = (a_{k_n})$ regarded as a series-to-series transformation. We call A^* the quasi method associated with A.

Given an infinite series† $\Sigma\, a_n$ and the matrix $A = (a_{nk})$, the quasi method A^* is to be 'applicable' if the A^*-transformation

$$b_n = \sum_{k=0}^{\infty} a_{kn}\, a_k \qquad \ldots(1)$$

exists for each $n \geqslant 0$. The series $\Sigma\, a_n$ is said to be A^* summable (or summable by the quasi-method associated with A) to the value s, if further

$$\Sigma\, b_n = s. \qquad \ldots(2)$$

When A is a normal matrix, then A^* is in general a row infinite matrix and therefore it is relatively more difficult to work with.

The generalised Nörlund matrix (N, p, q) defined by

$$a_{nk} = \begin{cases} \dfrac{p_{n-k}\, q_k}{r_n} & (k \geqslant n) \\ 0 & (k > n) \end{cases}$$

where $r_n = p_0\, q_n + \ldots + p_n\, q_0 \neq 0 \; (n > 0)$, is normal provided that $q_n \neq 0$. In the case $q_n = 1$ (all n) then the method reduces to the familiar Nörlund method (N, p), see Hardy[2] (p. 64).

Kuttner[3] defined quasi-Cesàro as quasi-Hausdorff transformation. Thorpe[7] has considered the transpose of the Nörlund matrix regarded as giving a series-to-series transformation in the case $p_n \in \mathcal{M}$, i.e.

† Summation without limits means from 0 to ∞.

$$p_n > 0, \frac{p_{n+1}}{p_n} \leqslant \frac{p_{n+2}}{p_{n+1}} \leqslant 1 \; (n = 0, 1, 2, \ldots).$$

Mohapatro[4,5] has considered the transpose of the generalised Nörlund matrix $(N, p. q)$ in the case $p_n \in \mathscr{M}$, $q_n > 0$, $q_{n+1} \geqslant q_n$.

2. Statlment of Theorems

Generalising the results of Thorpe[7] and Mohapatro[8] we obtain the inverse of quasi-matrix transformation in Theorem 1 below whereas in Theorem 2 we determine conditions under which it is possible to recapture the original transformation from its inverse transformation.

Before we state our theorems, we require the following notation :

We write $A \in \mathscr{P}$ if A is normal and satisfies

$$a_{nk} > 0 \; (k \leqslant n); \; \frac{a_{n+1,k+1}}{a_{n,k+1}} \leqslant \frac{a_{n+1,k}}{a_{nk}} \; (0 \leqslant k \leqslant n-1). \qquad \ldots(3)$$

We write $A \in \mathscr{P}^*$ if $A \in \mathscr{P}$ and

$$a_{nk} \leqslant a_{n,k+1} \; (0 \leqslant k \leqslant n-1). \qquad \ldots(4)$$

Theorem 1—Let $A \in \mathscr{P}$. Then the A^* transformation (when applicable) has an inverse whose matrix is given by the transpose of the inverse of the A transformation; that is, if b_n is given by (1) then

$$a_n = \sum_{k=n}^{\infty} a_{kn}^{-1} \, b_k. \qquad \ldots(5)$$

A matrix B, regarded as a series-to-series transformation, is called 'reversible,' if for any $b = (b_n)$ with $\Sigma \, b_n$ convergent, there is a 'unique' $a = (a_n)$ such that $Ba = b$. In the following theorem, we consider the reversibility of A^* transformation, regarded as a series-to series transformation.

Theorem 2—Let $A \in \mathscr{P}^*$. Then A^* transformation is reversible; more generally, if for any $b_n = o\,(1)$, then there exist a unique $a = (a_n)$ such that (1) holds.

For the proof of the theorem, we require the following lemmas :

Lemma 1 (a)—Let $A \in \mathscr{P}$. Then (see Peyerimhoff[6], p. 33)

(i) $a_{nn}^{-1} > 0, \; \forall \, n \geqslant 0; \; a_{kn}^{-1} \leqslant 0 \; (k > n)$;

(b) Let $A \in \mathscr{P}^*$. Then (see Das[1])

(ii) $\sum_{k=n}^{\infty} | a_{kn}^{-1} | < \infty$ for each $n \geqslant 0$,

(iii) $\sum_{k=n}^{\infty} a_{kn}^{-1} \geqslant 0$ for each $n \geqslant 0$,

(iv) $\sum_{k=m+1}^{\infty} | a_{kn}^{-1} | \leqslant \sum_{k=n}^{m} a_{kn}^{-1} \; (m \geqslant n)$.

Lemma 2—Let $A \in \mathscr{P}$. Then for $0 \leqslant n < N < M$, we have

$$0 \leqslant \sum_{k=n}^{N} a_{kn}^{-1} a_{Mk} \leqslant \frac{a_{Mn}}{a_{nn}} \leqslant \frac{a_{Mo}}{a_{no}}.$$

PROOF ; Since by Lemma 1 (1), $a_{nn}^{-1} > 0$ and $a_{kn}^{-1} \leqslant 0$ for $k > n$, it follows that

$$a_{nn}^{-1} a_{Mn} \geqslant \sum_{k=n}^{N} a_{kn}^{-1} a_{Mk} \geqslant \sum_{k=n}^{M} a_{kn}^{-1} a_{Mk} = 0$$

by use of the identity (6) as $M > n$. But by (3)

$$\frac{a_{nk}}{a_{nk-1}} \leqslant \frac{a_{mk}}{a_{mk-1}} \; (1 \leqslant k \leqslant m \leqslant n).$$

Hence it follows that

$$a_{nn}^{-1} a_{Mn} = \frac{a_{Mn}}{a_{nn}} = \frac{1}{a_{nn}} \cdot \frac{a_{Mn}}{a_{M-1}} \cdot \frac{a_{Mn-1}}{a_{Mn-2}} \cdots \frac{a_{M1}}{a_{Mo}} \cdot a_{Mo}$$

$$\leqslant \frac{1}{a_{nn}} \cdot \frac{a_{nn}}{a_{nn-1}} \cdot \frac{a_{nn-1}}{a_{nn-2}} \cdots \frac{a_{n1}}{a_{n0}} \cdot a_{M0}$$

$$= \frac{a_{M0}}{a_{n0}}.$$

This completes the proof of the lemma.

3. Proof of Theorem 1

Substituting the value of b_n as given in eqn. (1) and using the identity

$$\sum_{k=n}^{r} a_{kn}^{-1} a_{rk} = \begin{cases} 1 \; (r = n) \\ 0 \; (r > n) \end{cases} \qquad \ldots(6)$$

we have, for any integer $N > n$,

$$\sum_{k=n}^{N} a_{kn}^{-1} b_k = \sum_{k=n}^{N} a_{kn}^{-1} \sum_{r=k}^{\infty} a_{rk}\, a_r$$

$$= \sum_{k=n}^{N} a_{kn}^{-1} \left(\sum_{r=k}^{N} + \sum_{r=N+1}^{\infty} \right) a_{rk}\, a_r$$

$$= \sum_{r=n}^{N} a_r \sum_{k=n}^{r} a_{kn}^{-1} a_{rk} + \sum_{k=n}^{N} a_{kn}^{-1} \sum_{r=N+1}^{\infty} a_{rk}\, a_r$$

$$= a_n + \sum_{r=N+1}^{\infty} a_r \sum_{k=n}^{N} a_{kn}^{-1} a_{rk}. \qquad \text{...(7)}$$

So it is obvious from (7) that (5) holds if and only if, for fixed n,

$$\varphi_N = \sum_{r=N+1}^{\infty} a_r \sum_{k=n}^{N} a_{kn}^{-1} a_{rk} \to 0 \text{ as } N \to \infty. \qquad \text{...(8)}$$

Since $b_0 = \sum\limits_{k=0}^{\infty} a_{k0}\, a_k$ converges by hypothesis, writing

$$w_n = \sum_{k=n}^{\infty} a_{k0}\, a_k$$

we obtain

$$a_n = \frac{w_n - w_{n+1}}{a_{n0}}. \qquad \text{...(9)}$$

Substituting this value of a_n in (8), we have

$$\varphi_N = \sum_{r=N+1}^{\infty} \frac{w_r - w_{r+1}}{a_{r0}} \sum_{k=n}^{N} a_{kn}^{1} a_{rk}. \qquad \text{...(10)}$$

But for $M > N$

$$\sum_{r=N+1}^{M} \frac{w_r - w_{r+1}}{a_{r0}} \sum_{k=n}^{N} a_{kn}^{-1} a_{rk}$$

(equation continued on p. 1127)

$$= \sum_{r=N+1}^{M} w_r \sum_{k=n}^{N} a_{kn}^{-1} \left(\frac{a_{rk}}{a_{r0}} - \frac{a_{r-1,k}}{a_{r-1,0}} \right)$$

$$+ \frac{w_{N+1}}{a_{N0}} \sum_{k=n}^{N} a_{kn}^{-1} a_{Nk} - \frac{w_{M+1}}{a_{M0}} \sum_{k=n}^{N} a_{kn}^{-1} a_{Mk}. \qquad \ldots(11)$$

Since w_k is the remainder of a convergent series, it follows that $w_k = o\,(1)$ as $k \to \infty$. Hence by Lemma 2, for each fixed n and N

$$\frac{w_{M+1}}{a_{M0}} \sum_{k=n}^{N} a_{kn}^{-1} a_{Mk} = o\,(1)\, 0 \left(\frac{1}{a_{n0}} \right) = o\ (1)$$

as $M \to \infty$. Also by identity (6), as $N > n$,

$$\sum_{k=n}^{N} a_{kn}^{-1} a_{Nk} = 0.$$

Hence it follows from (10) and (11) that

$$\varphi_N = \sum_{r=N+1}^{\infty} w_r \sum_{k=n}^{N} a_{kn}^{-1} \left(\frac{a_{rk}}{a_{r0}} - \frac{a_{r-1,k}}{a_{r-1,0}} \right). \qquad \ldots(12)$$

Hence in order that $\varphi_N = o\,(1)$ as $N \to \infty$ for every (w_r) such that $w_r = o\,(1)$ as $r \to \infty$, it is necessary and sufficient that

$$\psi_N = \sum_{r=N+1}^{\infty} \left| \sum_{k=n}^{N} \left(\frac{a_{rk}}{a_{r0}} - \frac{a_{r-1,k}}{a_{r-1,0}} \right) a_{kn}^{-1} \right| = O\,(1) \qquad \ldots(13)$$

as $N \to \infty$, for fixed n.

By hypothesis (3) and Lemma 1 (a), the first term in the inner sigma in ψ_N is negative and all other terms for $k > n$ are non-negative. Hence the modulus of the inner sigma in (13) is not greater than

$$-\left(\frac{a_{rn}}{a_{r0}} - \frac{a_{r-1,n}}{a_{r-1,0}} \right) a_{nn}^{-1} + \sum_{r=n+1}^{N} \left(\frac{a_{rk}}{a_{r0}} - \frac{a_{r-1,k}}{a_{r-1,0}} \right) a_{kn}^{-1} \qquad \ldots(14)$$

and hence, it follows from (14) that

$$\psi_N \leqslant - \sum_{r=N+1}^{\infty} \left(\frac{a_{rn}}{a_{r0}} - \frac{a_{r-1,k}}{a_{r-1,0}} \right) a_{nn}^{-1}$$

$$+ \sum_{r=N+1}^{\infty} \sum_{k=n+1}^{N} \left(\frac{a_{rk}}{a_{r0}} - \frac{a_{r-1,k}}{a_{r-1,0}} \right) a_{kn}^{-1} . \qquad \ldots(15)$$

By hypothesis (3), the first expression in (15) is

$$a_{nn}^{-1} \left(\frac{a_{N,n}}{a_{N,0}} - \lim_{m\to\infty} \frac{a_{mn}}{a_{m0}} \right)$$

$$\leqslant a_{nn}^{-1} \frac{a_{nn}}{a_{n0}} = \text{constant.}$$

Since the terms in the second expression in (15) are non-negative, we may invert the order of summation and the double sum becomes

$$\sum_{k=n+1}^{N} a_{kn}^{-1} \sum_{r=N+1}^{\infty} \left(\frac{a_{rk}}{a_{r0}} - \frac{a_{r-1,k}}{a_{r-1,0}} \right)$$

$$= \sum_{k=n+1}^{N} a_{kn}^{-1} \left(\lim_{n\to\infty} \frac{a_{rk}}{a_{r0}} - \frac{a_{Nk}}{a_{N0}} \right)$$

$$\leqslant - \frac{1}{a_{N0}} \sum_{k=n+1}^{N} a_{kn}^{-1} a_{Nk} \quad \left(\text{since } a_{kn}^{-1} \leqslant 0 \text{ for } k > n \right)$$

$$= \frac{a_{Nn}\, a_{nn}^{-1}}{a_{N0}}$$

and this is bounded for fixed n.

This proves (13) and completes the proof of the theorem.

4. Proof of Theorem 2

Since $b_n = o(1)$, it follows from Lemma 1 (b) (ii) that (5) converges for each n; hence a_n is defined by (5).

Now by identity (6)

$$\sum_{\nu=n}^{N} a_{\nu n}\, a_{\nu} = \sum_{\nu=n}^{N} a_{\nu n} \sum_{k=\nu}^{\infty} a_{k\nu}^{-1}\, b_k$$

$$= \sum_{\nu=n}^{N} a_{\nu n} \left(\sum_{k=\nu}^{N} + \sum_{k=N+1}^{\infty} \right) a_{k\nu}^{-1}\, b_k$$

$$= b_n + \sum_{k=N+1}^{\infty} b_k \sum_{\nu=n}^{N} a_{\nu n}\, a_{k\nu}^{-1}\,.$$

Hence

$$\lim_{N\to\infty} \sum_{\nu=n}^{N} a_{\nu n}\, a_{\nu} = b_n$$

if and only, if, for fixed n,

$$\lim_{N\to\infty} \sum_{k=N+1}^{\infty} b_k \sum_{\nu=n}^{N} a_{\nu n}\;\; a_{k\nu}^{-1} = 0. \qquad \text{...(16)}$$

Since $b_k = o\,(1)$, to prove (16) it is enough to prove that, as $N \to \infty$, for fixed n,

$$\alpha_N = \sum_{k=N+1}^{\infty} \left| \sum_{\nu=n}^{N} a_{\nu n}\;\; a_{k\nu}^{-1} \right| = O\,(1). \qquad \text{...(17)}$$

But since by Lemma 1 (a), $a_{k\nu}^{-1} \leqslant 0 \; (k > \nu)$, all the terms in (17) have the same sign and so we may omit the modulus sign. Thus

$$\alpha_N = - \sum_{k=N+1}^{\infty} \sum_{\nu=n}^{N} a_{\nu n}\;\; a_{\nu k}^{-1}$$

$$= \sum_{\nu=n}^{N} a_{\nu n} \sum_{k=N+1}^{\infty} \left(- a_{k\nu}^{-1} \right)$$

$$\leqslant \sum_{\nu=n}^{N} a_{\nu n} \sum_{k=\nu}^{N} a_{k\nu}^{-1} = 1$$

by (iii) and (iv) of Lemma 1 (b).

This proves that (1) holds and the uniquenses of '*a*' follows from Theorem 1.

This completes the proof of Theorem 2.

Remark : Theorem 1 is unchanged if we multiply each row of A by a non-zero constant and each column by a non-zero constant; in other words, we replace (a_{nk}) in Theorem 1 by $(\lambda_n \mu_k a_{nk})$ where (λ_n) and (μ_n) are any sequences of non-zero constants.

For suppose that A satisfies the conditions of Theorem 1. Applying that theorem with a_n replaced by $\lambda_n a_n$ and b_n by b_n/μ_n we see that if

$$b_n = \mu_n \sum_{k=n}^{\infty} a_{kn} \lambda_k a_k$$

converges for all n, then

$$a_n = \lambda_n^{-1} \sum_{k=n}^{\infty} a_{kn}^{-1} \lambda_k^{-1} b_k.$$

Thus Theorem 1 applies rather more generally, to any matrix of the form $(\lambda_n \mu_k a_{nk})$ where (a_{nk}) satisfies the conditions in the form stated.

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References

1. G. Das, Modified means (under communication).
2. G. H. Hardy, *Divergent series*, Oxford, 1949.
3. B. Kuttner, *J. Indian Math. Soc.*, **24** (1960), 319–41.
4. P. Mahapatro, *Pacific J. Math.* **68** (1977), 177–95.
5. P. Mohaptro, *J. Indian Math. Soc.*, **43** (1979), 259–69.
6. A. Peyerimhoff, Lectures on summability, *Lecture Notes in Mathematics No.* 107, Springer-Verlag, Berlin.
7. B. Thorpe, Ph. D. Thesis. Birmingham University, 1970.
8. P. Vermes, *Colloque sur la Théorie des Suites*, *Centre Belge de recherches Math.*, (1958), 60-86.

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GENERALIZED STIELTJES TRANSFORM OF BANACH SPACE VALUED DISTRIBUTIONS

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In this paper an extension of a generalized Stieltjes transform

$$F(x) = \frac{\Gamma\, b\, \Gamma(\beta + 1)}{\Gamma(\alpha + b)}\, \frac{1}{x} \int_0^\infty \left(\frac{t}{x}\right)^\beta F\left(b, \beta + 1; b + \alpha; -\frac{t}{x}\right)$$

$$f(t)\, dt;$$

$$b = \beta + \eta + 1, \beta \geqslant 0, \eta > 0$$

to Banach space valued distributions (generalized functions) is provided. An inversion formula for the above transform is also proved in distributional (Banach space valued) sense.

1. INTRODUCTION

The theory of distributions has been worked out in great depth by L. Schwartz. Sebastiaoe Silva[1] also has a theory for vector valued distributions. Zemanian[2,3] has presented the theory of Banach space valued distributions. He has discussed the Laplace transform of Banach space valued distributions. Further he has used these concepts for applications in system theory and signals.

The author[4,5] extended the generalized Stieltjes transform, namely

$$F(x) = \frac{\Gamma(\beta + \eta + 1)\, \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + \eta + 1)}\, \frac{1}{x}\, .$$

$$\int_0^\infty \left(\frac{t}{x}\right)^\beta F\left(\beta + \eta + 1, \beta + 1; \alpha + \beta + \eta + 1; -\frac{t}{x}\right)$$

$$f(t)\, dt \qquad \ldots(1.1)$$

to distributions. The complex inversion formula for the transform (1.1) namely

$$f(t) = \frac{1}{2\pi i} \lim_{w \to \infty} \int_{c-wi}^{c+wi} \frac{\Gamma(d + s - \beta - 1)}{\Gamma(b + s - \beta - 1)\, \Gamma(\beta + 1 - s)\, \Gamma s}\, t^{-s}\, M(s)\, ds$$

where

$$M(s) = \int_0^\infty x^{s-1} F(x)\, dx$$

$$b = \beta + \eta + 1,\ d = b + \alpha,\ s = c + iw,\ \beta \geq 0,\ \eta > 0$$

was also extended to distributions by the author[4].

Motivated from the work of Zemanian[2,3] and Tiwari[4,5,6] we study generalized Stieltjes transform for Banach space valued distributions, in this paper.

The notation and terminology of this paper will follow that of Zemanian[2,3,7] and Tiwari[4,5,6] when U and V are two topological vector space, the symbol $[U; V]$ denotes the linear space of all continuous linear mappings of U into V. A and B will usually denote Banach spaces. Throughout the sequel we use the following notation :

$\mathbb{R}$ = set of all real numbers

$\mathbb{C}$ = set of all complex numbers

I = $(0, \infty)$.

$D(A)$ = space of smooth A-valued test functions with compact support defined on I.

2. The Spaces $S_a(A)$ and $[S_a(A); B]$

Definition 2.1—The space $S_a(A)$ is defined as the linear space of all smooth functions $\varphi(t)$ from I into A such that

$$\rho_{a,l}(\varphi) = \max_{0 \leq k \leq l} \sup_{t \in I} \| (1+t)^a t^k \varphi^k(t) \|_A$$

$$< \infty,\ l = 0, 1, 2, \ldots$$

The topology of $S_a(A)$ is generated by sequence of seminorms $\{\rho_{a,1}\}_{1=0}^{\infty}$. For $A = \mathbb{C}$ we write $S_a(\mathbb{C}) = S_a$. It is easy to see that $D(A) \subset S_a(A)$. If $\{a_v\}_{v=1}^{\infty}$ be a monotonic sequence of real numbers such that $a_v \to w+$, where w is a real number or $-\infty$, the space $S(w; A)$ is defined as

$$S(w; A) = \bigcup_{v=1}^{\infty} S_{a_v}(A).$$

For a Banach space B, any $f \in [S_a(A); B]$ will be called a Banach space valued distribution. When $A = B = \mathbb{C}$, f becomes a scalar distribution. The simple or weaker topology for $[S_a(A), B]$ is generated by the collection of seminorms $\{\gamma_\varphi\}_\varphi$, where φ traverses $S_a(A)$ and

$$\gamma_\varphi(f) = \| \langle f, \varphi \rangle \|_B.$$

We now come to the definition of the space $[S_a(A); B]$.

If $\varphi \in S_a$, $a \in A$, then $\varphi a = a\varphi$ is the function from I into A that assigns to each $t \in I$ the value $\varphi(t)\, a$. Clearly $\varphi a \in S_a(A)$. We denote by $S_a \otimes A$ the subspace of $S_a(A)$ consisting of elements of the form φa. If $g \in [S_a; \mathbb{C}]$ and $a \in A$, $ga = ag$ is defined by

$$<ga, \varphi> = <g, \varphi> a, \varphi \in S_a.$$

Clearly $ga \in [S_a; A]$. $[S_a; \mathbb{C}] \otimes A$ denotes the subspace of $[S_a; A]$ consisting of all linear combinations of elements of the form ga. Similarly when $g \in [S_a; [A; B]]$ and $a \in A$, we define $ga = ag \in [S_a; B]$. Every $g \in [S_a(A); B]$ uniquely defines an $f \in [S_a; [A; B]]$ through the equation

$$<f, \varphi> a = <g, \varphi a>, \ \varphi \in S_a, a \in A.$$

See Zemanian[2], p. 105.

3. Generalized Stieltjes Transform

It is proved in Tiwari[5] that

$$\frac{\Gamma b\, \Gamma(\beta+1)}{\Gamma d} \frac{1}{x} \left(\frac{t}{x}\right)^{\beta}$$

$$F\left(b, \beta+1; d; -\frac{t}{x}\right) \in S_a \text{ for } a \leqslant 1$$

where

$$b = \beta + \eta + 1, d = b + \alpha.$$

Now let $f \in [S_a, A]$, we define the generalized Stieltjes transform $F(x)$ of f by

$$F(x) = < f(t), \frac{\Gamma b\, \Gamma(\beta+1)}{\Gamma d} \frac{1}{x} \left(\frac{t}{x}\right)^{\beta} F\left(b, \beta+1; d; -\frac{t}{x}\right)>. \qquad \ldots(3.1)$$

The left hand side $F(x)$ is an A-valued function.

Similarly if $y \in [S_a(A); B]$. The generalized Stieltjes transform $Y(x)$ of y is defined as the generalized Stieltjes transform of f_y :

$$Y(x) = < f_y(t), \frac{\Gamma b\, \Gamma(\beta+1)}{\Gamma d} \frac{1}{x} \left(\frac{t}{x}\right)^{\beta} F\left(b, \beta+1; d; -\frac{t}{x}\right)> \qquad \ldots(3.2)$$

where

$$f_y \in [S_a; [A; B]].$$

The above definition is meaningful because corresponding to y we have the unique f_y as discussed in previous section. Note that $Y(x)$ is an $[A; B]$ valued function.

4. An Inversion Theorem

We need the following there lemmas for the proof of inversion theorem.

Lemma 4.1—Let $y \in [S_a(A); B]$, f_y be the corresponding member in $[S_a; [A; B]]$ and

$$\theta(x, \tau) = \frac{\Gamma b \, \Gamma(\beta+1)}{\Gamma d} \frac{1}{x} \left(\frac{\tau}{x}\right)^{\beta} F\left(b, \beta+1; d; -\frac{\tau}{x}\right).$$

Then

$$\int_0^\infty x^{-s} < y(\tau), \theta(x, \tau) > dx$$

$$= < y(\tau), \int_0^\infty \theta(x, \tau) \, x^{-s} \, dx >.$$

Lemma 4.2—Let $y \in [S_a(A), B]$, $\varphi \in D(A)$ and

$$P(s) = \int_0^\infty \varphi(t) \, t^{-s} \, dt.$$

Then, for any two fixed real numbers r and c such that $0 < r < \infty$, $s = c + iw$

$$\int_{-r}^{r} < f_y(\tau), \tau^{s-1} > P(s) \, dw$$

$$= < y(\tau), \int_{-r}^{r} \tau^{s-1} P(s) \, dw >.$$

Lemma 4.3—If $\varphi \in D(A)$ and a, c, r be real numbers with $a < c$. Then

$$\frac{1}{\pi} \int_0^\infty \frac{\varphi(v)}{u \log\left(\frac{u}{v}\right)} \left(\frac{u}{v}\right)^c \sin\left(r \log \frac{u}{v}\right) dv$$

converges in $S_a(A)$ to $\varphi(u)$ as $r \to \infty$.

Except for some obviuos changes proofs of the above lemmas are respectively similar to those of Lemmas 4.1, 4.2 and 4.3 in Tiwari[4]. We now come to inversion theorem.

Theorem 4.1—If $y \in [D(A); B]$ and $Y(x)$ be the generalized Stieltjes transform of Banach space valued distribution y, then in the sense of weak convergence in $[D(A); B]$.

$$\lim_{r\to\infty} < \frac{1}{2\pi i}\int_{c-ir}^{c+ir} \frac{\Gamma(d+s-\beta-1)}{\Gamma(b+s-\beta-1)\,\Gamma(\beta+1-s)\,\Gamma(s)} M(s)\, t^{-s}\, ds, \varphi(t)>$$

$$= <y, \varphi>$$

where

$$M = \int_0^\infty x^{s-1}\, y(x)\, dx.$$

Proof : On substituting

$$\frac{\Gamma(d+s-\beta-1)}{\Gamma(b+s-\beta-1)\,\Gamma(\beta+1-s)\,\Gamma s} = Q$$

and

$$s = c + iw$$

we have

$$< \frac{1}{2\pi i}\int_{c-ir}^{c+ir} \frac{\Gamma(d+s-\beta)}{\Gamma(b+s-\beta-1)\,\Gamma(\beta+1-s)\,\Gamma s} M(s)\, t^{-s}\, ds, \varphi(t)>$$

$$= < \frac{1}{2\pi}\int_{-r}^{r} Q M(s)\, t^{-s}, \varphi(t)>. \qquad \ldots(4.1)$$

The integral on w is an $[A; B]$ valued continuous function of t. By the continuity of the B-valued function $Y(x)\,\varphi(t)$ on the $w-t$ plane and using change of order of integration (4.1)

$$= \frac{1}{2\pi}\int_{-r}^{r} Q\, M(s) \int_0^\infty t^{-s}\,\varphi(t)\, dt\, dw$$

$$= \frac{1}{2\pi}\int_{-r}^{r} Q \int_0^\infty x^{s-1}\, y(x)\, dx \int_0^\infty t^{-s}\,\varphi(t)\, dt\, dw$$

$$= \frac{1}{2\pi}\int_{-r}^{r} Q \int_0^\infty x^{s-1} <y(\tau), \frac{\Gamma b\,\Gamma(\beta+1)}{\Gamma d}\,\frac{1}{x}\,\left(\frac{\tau}{x}\right)^{\beta}.$$

(equation continued on p. 1136)

$$F\left(b, \beta + 1; d; -\frac{\tau}{x}\right) > dx \int_0^{\infty} t^{-s} \varphi(t)\, dt\, dw \qquad \text{...(4.2)}$$

$$= \frac{1}{2\pi} \int_{-r}^{r} Q < y(\tau), \int_0^{\infty} x^{s-1} \frac{\Gamma b\, \Gamma(\beta+1)}{\Gamma d} \frac{1}{x} \left(\frac{\tau}{x}\right)^{\beta}$$

$$\times F\left(b, \beta + 1; d; -\frac{\tau}{x}\right) dx > \int_0^{\infty} t^{-s} \varphi(t)\, dt\, dw \qquad \text{...(4.3)}$$

by Lemma (4.1).

Using the result of Tiwari[4] (p. 1052), viz.

$$Q \int_0^{\infty} x^{s-1} \int_0^{\infty} \frac{\Gamma b\, \Gamma(\beta+1)}{\Gamma d} \frac{1}{x} \left(\frac{\tau}{x}\right)^{\beta} F\left(b, \beta + 1; d; -\frac{\tau}{x}\right)$$

$$\times y(\tau)\, d\tau\, dx$$

$$= \int_0^{\infty} \tau^{s-1} y(\tau)\, d\tau$$

we have

$$(4.3) = \frac{1}{2\pi} \int_{-r}^{r} < y(\tau), \tau^{s-1} > \int_3^{\infty} t^{-s} \varphi(t)\, dt\, dw$$

$$= < y(\tau), \frac{1}{2\pi} \int_{-r}^{r} \tau^{s-1} \int_0^{\infty} t^{-s} \varphi(t)\, dt\, dw > \qquad \text{...(4.4)}$$

by Lemma (4.2).

The proof now can be completed as in Tiwari[4] (p. 1053).

References

1. J. Sebasiao e Silva, *Portugal Math.*, **19** (1960), 1–80.
2. A. H. Zemanian, *SIAM J. appl. Math.* **18** (1970), 98–138.
3. A. H. Zemanian, *IEEE Trans. Circuits and Systems*, *CAS*-25 **9** (1978), 793–99.
4. A. K. Tiwari, *Indian J. pure appl. Math.* **11** (1980), 1045–54.
5. A. K. Tiwari, and P. S. Koranne, *Indian J. pure appl. Math.* 16 (1985), 383–94.
6. A. K. Tiwari, and R. K. Goyal, *J. Inst. Electronics, Telec. Engrs*, Communicated.
7. A. H. Zemanian, *Gneralized Integral Transformations*. Interscience Pub., New York, 1968.

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ON SINGLETONNESS OF UNIQUELY REMOTAL SETS

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It is shown that a bounded subset G of a finite dimensional Hilbert space H, whose closure, $\bar{G}$, admits a unique farthest point to its Chebyshev center is necessarily a singleton. The same result also holds when H is infinite dimensional and G is nearly compact.

For a non-empty bounded subset G of a normed linear space X, the farthest distance function from G is defined by

$$t(x) = \sup_{a \in G} \|x - a\|, \ (x \in X).$$

The subset G is said to be uniquely remotal (or to have the unique farthest point property) in X if for each $x \in X$ there exists a unique element $q(x) \in G$ such that $t(x) = \|x - q(x)\|$. The following result is known for finite dimensional Hilbert spaces[2].

Proposition—Every uniquely remotal subset G of a finite dimensional Hilbert space is necessarily a singleton.

We also know[3] that the unique farthest point property of a uniquely remotal subset G of a strictly convex normed linear space is inherited by $\bar{G}$, the norm closure of G. Knowing this, and the fact that Hilbert spaces are strictly convex, we can (equivalently) restate the above proposition as follows :

Theorem —Let G be a subset of a finite dimensional Hilbert space H, such that $\bar{G}$ admits a unique farthest point to each $x \in H$. Then G is necessarily a singleton.

We now strengthen the above Theorem, as the following main result of this note. Recall that every bounded subset G of a Hilbert space has a unique Chebyshev center c (see Astaneh[1], p. 1312); i. e., there exists a unique $c \in H$ such that $t(c) = \inf_{x \in H} t(x)$. The non-negative real number $t(c)$ is called the Chebyshev radius of G and is denoted by $r(G)$.

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Theorem 1—Let G be a bounded subset of a finite dimensional (real or complex) Hilbert space H. If $\bar{G}$ admits a unique farthest point to the Chebyshev center c of G, then G is necessarily a singleton.

Proof : Suppose G is not a singleton. Without any loss of generality we may assume that $r(G) = 1$ and $c = 0$ is the Chebyshev center of G. Let x_0 denote the unique farthest point for $c = 0$ in $\bar{G}$. Then x_0 is the only point in $\bar{G} \cap U(H)$ with $\|x_0\| = 1$. (Here $U(H)$ is the closed unit ball of H). Consider the sequence $x_n = x_0/n$, $(n > 2)$ in the open line segment $]0, x_0[$, and note that for each n

$$1 < t(x_n)^2 = \sup_{a \in G} \|x_n - a\|^2.$$

(The strict inequality on the left is due to the uniqueness of the center 0 of G, which does not lie on $]0, x_0[$). Therefore, for each n we can choose $a_n \in G$ such that

$$1 < \|x_n - a_n\|^2 \leqslant t(x_n)^2. \qquad \ldots(1)$$

Clearly for each n we have $\|a_n\| < 1$, and hence by (1),

$$1 > \|a_n\|^2 = \|x_n - a_n\|^2 - \|x_n\|^2 + 2\,\mathrm{Re} <x_n, a_n>$$

$$= \|x_n - a_n\|^2 - \frac{1}{n^2} + \left(\frac{2}{n}\right) \mathrm{Re} <x_0, a_n>$$

$$> 1 - \frac{1}{n^2} + \left(\frac{2}{n}\right) \mathrm{Re} <x_0, a_n>.$$

Therefore, for each n the following inequality holds

$$2\mathrm{Re} <x_0, a_n> < \frac{1}{n}. \qquad \ldots(2)$$

Considering (1) and (2), for each n we have

$$\|x_0 - a_n\|^2 = \| x_0 - x_n \|^2 + \|x_n - a_n\|^2 + 2\,\mathrm{Re} <x_0 - x_n, x_n - a_n>$$

$$= \left(1 - \frac{1}{n}\right)^2 + \|x_n - a_n\|^2 + 2\left(1 - \frac{1}{n}\right)$$

$$\mathrm{Re} <x_0, x_n - a_n>$$

$$= \left(1 - \frac{1}{n}\right)^2 + \|x_n - a_n\|^2 + 2\left(1 - \frac{1}{n}\right)\frac{1}{n}$$

$$- 2\left(1 - \frac{1}{n}\right) \mathrm{Re} <x_0, a_n>$$

$$> \left(1 - \frac{1}{n}\right)^2 + 1 + 2\left(1 - \frac{1}{n}\right)\frac{1}{n} - \left(1 - \frac{1}{n}\right)$$

$$\frac{1}{n} = 2 - \frac{1}{n}.$$

Therefore, for each $n > 2$ we get

$$\|x_0 - a_n\| > \sqrt{\frac{3}{2}}. \qquad \ldots(3)$$

On the other hand since the farthest distance function $x \to t(x)$ is continuous, and since $x_n \to 0$ we have $\underset{n}{\text{limit}}\,(x_n) = t(0) = 1$. From this and (1) it follows that

$\|x_n - a_n\| \to 1$. Next from $\|x_n - a_n\| \to 1$ and

$$\|x_n - a_n\| - \frac{1}{n} = \|x_n - a_n\| - \|x_n\| \leqslant \|a_n\| < 1$$

we deduce that $\lim_n \|a_n\| = 1$.

This means that $\{a_n\}$ is a maximizing sequence for 0 in G. We now observe that, H being finite dimensional, $\{a_n\}$ has a cluster point a_0 which of course lies in $\bar{G}$. Since $\|a_0\| = 1$ and $\|x_0 - a_0\| \geqslant \sqrt{\frac{3}{2}}$ (by (3)), it follows that 0 has another farthest point a_0 in $\bar{G}$ distinct from x_0. This contradiction completes the proof of the theorem.

We observe that the finite dimensionality assumption on H in the above proof is not used until the last paragraph, where we require a cluster point for the maximizing sequence $\{a_n\}$ for 0 in G. This observation enables us to deduce the following theorem in the infinite dimensional case, as well.

Theorem 2—Let G be a bounded subset of an infinite dimensional (real or complex) Hilbert space H. Let $\bar{G}$ admit a unique farthest point to Chebyshev center c of G, and assume that every maximizing sequence for c in G has a cluster point (i. e. G is nearly compact). Then G is necessarily a singleton.

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REFERENCES

1. A. A. Astaneh, *Indian J. pure appl. Math.*, **14** (1983), 1311-17.
2. N. Klee, *Math. Ann.* **142** (1961), 292-304.
3. T. D. Narang, *Nieuw. Arch. Voor Wisk.* (3), **25** (1977), 54-79.

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ON THE COEFFICIENTS OF POWERS OF A CLASS OF BAZILEVIC FUNCTIONS

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Sharp estimates for coefficients of powers of a class of Bazilevic functions are given and the smallest subclass, yet known, of univalent functions, for which the powers of the Koebe function are not extremal, is introduced.

INTRODUCTION

Suppose that $0 < \alpha < \infty$. For functions $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ analytic and univalent in the open unit disk $U = \{z: |z| < 1\}$ and for the Koebe function $k(z) = z(1 - z)^{-2}$ we write

$$\left\{\frac{f(z)}{z}\right\}^{1/\alpha} = \sum_{n=0}^{\infty} a_n(\alpha) z^n \text{ and } \left\{\frac{k(z)}{z}\right\}^{1/\alpha} = \sum_{n=0}^{\infty} b_n(\alpha) z^n \qquad \ldots(1)$$

so that

$$a_1(\alpha) = \frac{1}{\alpha} a_2,\ a_2(\alpha) = \frac{1}{\alpha}\left[a_3 - \frac{\alpha - 1}{2\alpha} a_2^2\right] \text{and}$$

$$b_n(\alpha) = \frac{2(2 + \alpha)(2 + 2\alpha) \ldots (2 + (n - 1)\alpha)}{(n!)\alpha^n}. \qquad \ldots(2)$$

We consider the inequality

$$|a_n(\alpha)| \leqslant b_n(\alpha). \qquad \ldots(3)$$

If $\alpha = 1$, this is the Bieberbach conjecture[3] which has recently been proved by De Branges[4] and for $\alpha = 2$ it is the Littlewood-Paley[10] conjecture which was disproved by Fekete-Szegö[5].

In fact (3) is true for univalent functions if $\alpha \leqslant 1$ and is false for $\alpha > 1$ and $n = 2$. (See Hayman and Hummel[6]).

We may ask whether (3) is still true for subclasses of univalent functions. In this paper we introduce the smallest class, yet known, of univalent functions for which the

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inequality (3) is false. Much of the work of this paper is drawn from the author's D. Phil. thesis[7].

For $\beta > 0$ let $B(\beta)$, called Bazilevic of type β, denote the class of functions $f(z)$ analytic in U satisfying the differential equation

$$zf'(z)(\{f(z)\}^{\beta-1} = (\{g(z)\}^{\beta}\, p(z) \qquad \ldots(4)$$

where g is a normalized starlike function so that

$$\frac{zg'(z)}{g(z)} = q(z). \qquad \ldots(5)$$

Here $p(z)$ and $q(z)$ are functions of positive real part with $p(0) = q(0) = 1$. It has been proved by many authors including Bazilevic[2] that the functions in $B(\beta)$ are univalent. (See also Sheil-Small[11,12]).

Note that $B(1)$ is the class of normalised close-to-convex functions defined by Kaplan[8].

Theorem 1—Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $B(1)$. Then we have the sharp bounds

$$|a_2(\alpha)| \leqslant \begin{cases} \dfrac{2+\alpha}{\alpha^2} & \text{if } 0 < \alpha \leqslant 3 \\[2ex] \dfrac{11\alpha - 3}{9\alpha(\alpha-1)} & \text{if } 3 < \alpha < \infty. \end{cases}$$

Corollary 1—For $3 < \alpha < \infty$, Theorem 1 shows that the inequality (3) does not hold for powers of functions in $B(1)$.

Baernstein[1] proved that if $f(z) = z + \ldots$ is univalent, then for $h_4(z) = \{f(z^4)\}^{1/4} = z + \sum_{n=1}^{\infty} b_{4n+1} z^{4n+1}$, $|b_{4n+1}| \leqslant Kn^{-1/2}$ where K is an absolute constant. Keogh and Miller[9] obtained tke estimate $|b_{mn+1}| \leqslant \left[\frac{n-1+2/m}{n}\right]$ for $h_m(z) = z + \sum_{n=1}^{\infty} b_{mn+1} z^{mn+1}$ if $h_m \in B(\beta)$ and $0 < \beta \leqslant 1$. Little is known about the coefficients of $h_m(z)$ in $B(m)$ when m is an integer and $m \geqslant 2$. We prove that

Theorem 2—Suppose that $h_m(z) = z + \sum_{n=1}^{\infty} b_{mn+1} z^{mn+1}$ belongs to $B(m)$ where m is a positive integer. Then we have the sharp bounds

$$|b_{2m+1}| \leqslant \begin{cases} \dfrac{2+m}{m^2} & \text{if } m \leqslant 3 \\[2ex] \dfrac{11m - 3}{9m(m-1)} & \text{if } m \geqslant 4. \end{cases}$$

Corollary 2—For $m = 2$, Theorem 2 shows that Fekete-Szegö's[5] disproof of the Littlewood-Paley[10] conjecture does not apply to the square-root transform of close-to-convex functions.

Theorem 3—If m is an integer, $m \geqslant 3$, consider the meromorphic univalent function $g(\xi) = \{f(z^m)\}^{-1/m} = \xi + c_{m-1}\,\xi^{1-m} + c_{2m-1}\,\xi^{1-2m} + \ldots$

where $f(z) \in B(1)$, $z = 1/\xi$ and $|\xi| > 1$. Then we have the sharp bound

$$\left|c_{2m-1}\right| \leqslant \frac{11m+3}{9m(m+1)}\,.$$

If $m = 1$, the classical estimate $|c_1| \leqslant 1$ is sharp. The case $m = 2$ is more difficult and we omit it.

The estimate given in Theorem 3 corresponds to the estimate $|c_{2m-1}| \leqslant m^{-1}\{1 + 2e^{-2(m+1)/(m-1)})$ obtained by Fekete-Szegö[5] for the whole class of m-fold symmetric meromorphic univalent functions.

To prove our theorems we shall need the following lemma.

Lemma 1—Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ be such that $\operatorname{Re} p(z) > 0$ in $|z| < 1$. Then $\left|p_2 + \alpha p_1^2\right| \leqslant 2 + \alpha |p_1|^2$ if $\alpha \geqslant -1/2$.

PROOF : Write $p(z) = \{1 + \omega(z)\}/\{1 - \omega(z)\}$ where $\omega(z) = w_1 z + w_2 z^2 + \ldots$ and $|\omega(z)| \leqslant |z < 1$. Equate the coefficients of z and z^2 to obtain $p_1 = 2w_1$ and $p_2 = 2\left(w_2 + w_1^2\right)$. By applying the fact that $|w_2| \leqslant 1 - |w_1|^2$ we obtain $|p_2 - 1/2\, p_1^2| \leqslant 2 - 1/2\, |p_1|^2$. Now for $\alpha \geqslant -1/2$ add $(\alpha + \frac{1}{2})\,|p_1|^2$ to the both sides of the latter inequality to obtain the lemma.

Proof of Theorem 1—We see that the coefficients of functions $f(z)$ in $B(1)$ can be expressed by means of (4) and (5) in terms of the coefficients of two positive real part functions $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and $q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n$ and so can the coefficients $a_n(\alpha)$ of $\{z^{-1} f(z)\}^{1/\alpha}$ defined by (1).

Therefore we obtain $a_1(\alpha) = (p_1 + q_1)/(2\alpha)$ and

$$a_2(\alpha) = \frac{1}{\alpha}\left\{\frac{1}{6}q_2 + \frac{1}{3}p_2 + \frac{\alpha+3}{24\alpha}q_1^2 - \frac{\alpha-1}{8\alpha}p_1^2 + \frac{\alpha+3}{12\alpha}p_1q_1\right\}. \qquad \ldots(6)$$

It is easy to see that $|a_1(\alpha)| \leqslant 2/\alpha$ since $|p_1| \leqslant 2$ and $|q_1| \leqslant 2$.

For $a_2(\alpha)$ we obtain

$$\alpha\left|a_2(\alpha)\right| \leqslant \frac{1}{6}\left|q_2 + \frac{\alpha+3}{4\alpha}q_1^2\right| + \frac{1}{3}\left|p_2 - \frac{3(\alpha-1)}{8\alpha}p_1^2\right|$$

$$+ \frac{\alpha+3}{12\alpha}\left|p_1 q_1\right|$$

and so by Lemma 1 it follows that

$$\alpha \mid a_2(\alpha) \mid \leqslant \frac{1-\alpha}{8\alpha} \mid p_1 \mid^2 + \frac{\alpha+3}{6\alpha} \mid p_1 \mid + \frac{7\alpha+3}{6\alpha} = R(\mid p_1 \mid).$$

We require the maximum of $R(\mid p_1 \mid)$ when $0 \leqslant \mid p_1 \mid \leqslant 2$ and α is fixed and positive. We note that $R(0) = (7\alpha+3)/(6\alpha)$ and $R(2) = (\alpha+2)/\alpha = \alpha b_2(\alpha)$. Thus the maximum of $a_2(\alpha)$ is greater than $b_2(\alpha)$ if and only if $R(x)$ attains its maximum at p_1^0 in the interval (0.2). Now from

$$\frac{d}{d\mid p_1\mid}R(\mid p_1 \mid) = \frac{1-\alpha}{4\alpha}\mid p_1 \mid + \frac{\alpha+3}{6\alpha} = 0.$$

We obtain

$$\mid p_1^0 \mid = \{2(\alpha+3)\}/\{3(\alpha-1)\} \text{ and so } R(\mid p_1^0 \mid) = (11\alpha-3)/\{9(\alpha-1)\}.$$

We observe that $R(\mid p_1^0 \mid) > \max\{R(0), R(2)\}$ if $3 < \alpha < \infty$ and $R(2) > R(0)$ if $0 < \alpha \leqslant 3$ since $\mid p_1^0 \mid > 2$ when $0 < \alpha < 3$. This yields the bounds in Theorem 1. To show that they are sharp we write $q_1 = q_2 = p_2 = 2$ and $p_1 = 2$ for $0 \leqslant \alpha \leqslant 3$, $p_1 = \{2(\alpha+3)\})/(3(\alpha-1)\}$ if $\alpha > 3$. This corresponds to $p(z) = (1+\omega)/(1-\omega)$ where $\omega = z(p_1+2z)/(2+p_1 z)$ and $q(z) = (1+z)/(1-z)$ and hence $f(z)$ is constructed from (4) and (5).

To prove Theorem 2 we shall need the following well-known lemma. (See also Sheil-Small[11], p. 142).

Lemma 2—Suppose that n is a positive integer. Then $f(z) \in B(\beta)$ if and only if $\{f(z^n)\}^{1/n} \in B(n\beta)$.

Proof of Theorem 2—For $h_m(z) = \{f(z^m)\}^{1/m} \in B(m)$ we obtain by Lemma 2 that $f(z) \in B(1)$. Since for $f(z) \in B(1)$ both functions $\{z^{-1} f(z)\}^{1/m}$ and $\{z^{-m} f(z^m)\}^{1/m}$ have the same coefficients which are defined by (1) for $m = \alpha$, we apply Theorem 1 to $f(z)$ noting that $b_{2m+1} = a_2(m)$. Therefore we deduce Theorem 2.

Proof of Theorem 3—Since $f(z) \in B(1)$, its coefficients can be expressed in terms of the coefficients of two positive real part functions $p(z) = 1 + p_1 z + \ldots$ and

$q(z) = 1 + q_1 z + \ldots$ and so can the coefficients of $g(\xi) = \{f(z^m)\}^{-1/m}$. In fact with the notation of (1) we have $c_{2m-1} = a_2(-m)$. Therefore by applying (6) we obtain

$$-m\, c_{2m-1} = \frac{1}{6} q_2 + \frac{1}{3} p_2 + \frac{m-3}{24m} q_1^2 - \frac{m+1}{8m} p_1^2$$

$$+ \frac{m-3}{12m} p_1 q_1.$$

Thus if $m \geqslant 3$ we obtain

$$m \left| c_{2m-1} \right| \leqslant \frac{1}{6} | q_2 | + \frac{m-3}{24m} | q_1 |^2 + \frac{m-3}{12m} | p_1 q_1 |$$

$$+ \frac{1}{3} \left| p_2 - \frac{3(m+1)}{8m} p_1^2 \right|.$$

Since $1/2 \geqslant 3(m+1)/8m$, by Lemma 1 we deduce that

$$m \left| c_{2m-1} \right| \leqslant - \frac{m+1}{8m} | p_1 |^2 + \frac{m-3}{6m} | p_1 | + \frac{7m-3}{6m}$$

$$= R(| P_1 |).$$

We observe that $R(0) = (7m - 3)/(6m)$ and $R(2) = (m - 2)/m$. Also from

$$\frac{d}{d\,| p_1 |} R(| p_1 |) = - \frac{m+1}{4m} | p_1 | + \frac{m-3}{6m} = 0$$

we obtain $| p_1^0 | = (2m - 6)/(3(m+1)\}$. Now it is easy to see that $R(| p_1 |)$ has a maximum at $| p_1^0 |$ because for $m \geqslant 3$, $R(| p_1^0 |) = (11m + 3)/\{(9(m+1)\} \geqslant \max\{R(0), R(2)\}$ and so the desired result follows.

References

1. A. Baernstein II, Coefficients of univalent functions with restricted maximum modulus. To appear.
2. I. E. Bazilevic, *Math. Sb. N. S.* 37 (1955), 471–76.
3. L. Bieberbach, Uber die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, *Sitzungsberichte preussische Akademie der Wissenschaften* (1916), 940–55.
4. L. De Branges, *Acta Math.* 145 (1985), 137–52.
5. M. Fekete, and G. Szegö, *J. London Math. Soc.* 8 (1933) 85–89.
6. W. K. Hayman, and J. A. Hummel, Coefficients of powers of univalent functions, Complex Variables, To appear.
7. M. Jahangiri, Ph. D. Thesis, University of York, Great Britain 1985.
8. W. Kaplan, *Mich. Math. J.* 1 (1952), 169–85.
9. F. R. Keogh, and S. S. Miller, *Proc. Am. Math. Soc.* 30 (1971), 492–96.
10. J. E. Littlewood, and R. E. A. C. Paley, *J. London Math. Soc.* 7 (1932), 167-69.
11. T. B. Sheil-Small. *Quart. J. Math.* (2) 23, (1972) 135–42.
12. T. B. Sheil-Small, *J. Analyse Math.* 43 (1983/84), 1–11.

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FREE CONVECTION FLOW ON A NONISOTHERMAL FLAT PLATE AS INFLUENCED BY SUCTION OR BLOWING UNDER NONUNIFORM GRAVITY

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The effect of blowing and suction on free convection boundary layers developed on a nonisothermal flat plate under nonuniform gravity are considered. A numerical solution is obtained for various values of poroisty parameter F_w and temperature distributions. The skin-friction and heat transfer rate on the wall at selected values of the distance from the leading edge and for some values of F_w are presented.

1. INTRODUCTION

Previous work on the effects of suction and blowing on free convection boundary layers has been confined almost entirely to the case of a heated vertical plate. Eichhorn[1] considered the power law variation in plate temperature and transpiration velocity which yields a similarity solution of the boundary-layer equations. Sparrow and Cess[2] discussed the case of constant plate temperature and transpiration velocity. They[2] obtained series expansions for temperature and velocity in powers of $x^{1/4}$ where x is the distance measured from the leading edge. Merkin[3] extended the problem by obtaining asymptotic expansions, i. e. as $x \to \infty$, for temperature and velocity as influenced by both the blowing and the suction. The series expansions for small j given in Sparrow and Cess[2] were then joined to the asymptotic solution[3] by a numerical solution of the boundary-layer equations.

In a recent paper, Clarke[4] expanded the analysis presented in Eichhorn[1] by obtaining further approximations to the solution of the full Navier-Stokes equations for large, but finite, Grashof number. In ref. [4] the density variations were considered, where as in Eiechhorn[1], Sparrow and Cess[2], Merkin[3] the density variations were assumed important only in producing the buoyancy force.

Kao *et al.*[5] presented a solution for free convection along a vertical plate with arbitrary wall temperature variations in the absence of mass transfer at the wall. Aroesty *et al.*[6] and Merkin[7] considered the effect of blowing and suction through a body having a general shape.

The purpose of this paper is to consider the effects of blowing and suction on the steady nonisothermal laminar free convection flows along (i) an infinite cold porous

plate rotating at speed ω rads^{-1} in a radial plane with its leading edge being at a distance x_0 from the axis of rotation, and (ii) a finite hot porous plate of length x_0, rotating at speed ω rads^{-1} in a radial plane about the line $x = 0$, assuming the gravity varies linearly with the distance x. The temperature distributions over the plate are also assumed to be polynomial of first and second degree, respectively. Numerical solutions were adopted and results for the skin-friction and heat transfer rate are presented for different poroisty parameter F_w (−0.43, −0.2. 0.0, 0.2 and 0.43) with Prandtl numbers Pr 0.7 and 1.0.

The analogous problem of a free convection boundary layer on a vertical impermeable flat pate under nonuniform gravity has been treated by, Pop and Na[8], and similar cases to those given in Pop and Na[8] are considered to solve the present problem.

2. Problem Formulation

In the present analysis, an infinite cold plate rotating at speed ω rads^{-1} in a radial plane with its leading edge beginning at a distance x_0 from the axis of rotations, and a finite hot plate of length x_0, rotating at speed ω rads^{-1} in a radial plane about the line $x = 0$ subject to a nonuniform gravity field $g(x)$ are considered. The main effort in the present study is confined to the cases where the plate is porous and the plates temperature distributions $T_w(x)$ have specific forms.

Let the coordinates be chosen such that the x-coordinate is equal to the distance measured from the leading edge of the plate and the y-coordinate is the distance measured along the direction normal to the plate. The flow is assumed to be steady and incompressible. The boundary layer approximations are adopted along with the requirement that $\beta(T_w - T_\infty) \ll 1$ (large Taylor number). Thus the equations governing the conservation of mass, momentum and energy can be written in a dimensionless from as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \ldots(1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g(x)\, S_w(x)\, \theta + \frac{\partial^2 u}{\partial y^2} \qquad \ldots(2)$$

$$u\left[\frac{\partial \theta}{\partial x} + \theta\frac{d \ln S_w(x)}{dx}\right] + v\frac{\partial \theta}{\partial y} = \frac{1}{P_r}\frac{\partial^2 \theta}{\partial y^2} \qquad \ldots(3)$$

with the boundary conditions

$$\left.\begin{aligned} &y = 0,\ u = 0,\ v = \pm v_w(x),\ \theta = 1 \\ &y \to \infty, u = 0, \theta = 0. \end{aligned}\right] \qquad \ldots(4)$$

Here u and v are the velocity components along the x, y coordinates, $S_w(x)$ is the wall temperature, θ the temperature, Pr the Prandtl number and $v_w(x)$ with minus sign corresponds to suction and with plus sign — to injection'

Using the coordinate transformation

$$\xi = x, \; \eta = \frac{S_w (x)^{1/4}}{(4x)^{1/4}} y \qquad \ldots(5)$$

and introducing new dependent variables F and G such that

$$F(\xi,\eta) = \frac{\psi}{(64)^{1/4} x^{3/4} S_w (x)^{1/4}}, \; G(\xi,\eta) = \theta. \qquad \ldots(6)$$

These transformations reduce the boundary-layer equations to:

$$F''' + [3 + P(\xi)] FF'' - 2[1 + P(\xi)] F'^2 + g(\xi) G$$

$$= 4\xi \left(F' \frac{\partial F'}{\partial \xi} - F'' \frac{\partial F}{\partial \xi} \right) \qquad \ldots(7)$$

$$\frac{1}{Pr} G'' + [3 + P(\xi)] FG' - 4P(\xi) F' G = 4\xi \left(F' \frac{\partial G}{\partial \xi} - G' \frac{\partial F}{\partial \xi} \right) \qquad \ldots(8)$$

with the boundary conditions

$$\left. \begin{aligned} &F'(\xi, 0) = \mp F_w = \text{const.}, \; F'(\xi, 0) = 0, \; G(\xi, 0) = 1 \\ &F^{-}(\xi, 0) = 0, \; G(\xi, \infty) = 0 \end{aligned} \right] \qquad \ldots(9)$$

where the primes denote differentiation with respect to η, and the function $P(\xi)$ of the variable surface temperature is defined by

$$P(\xi) = \frac{\xi}{S_w(\xi)} \frac{dS_w}{d\xi}. \qquad \ldots(10)$$

The physical quantities of primary interest are the local skin-friction coefficient C_f and the local Nusselt number Nu which can be written respectively, in the form

$$C_f = \frac{\tau_w}{\rho \left(\frac{\nu}{u} \right)^2}, \; Nu = \frac{h_w x}{k}. \qquad \ldots(11)$$

From the definition of wall skin-friction $\tau_w = \mu \left(\frac{\partial u}{\partial y} \right) |_{y=0}$ and the local heat transfer coefficient $h_w = q_w/(T_w - T_\infty)$, where $q_w = -k (\partial T/\partial y)_{y=0}$, C_f and Nu are expressed as follows

$$C_f = 4 \left(\frac{Gr_x}{4} \right)^{3/4} F''(\xi, 0),$$

$$Nu = - \left(\frac{Gr_x}{4} \right)^{1/4} G'(\xi, 0). \qquad \ldots(12)$$

3. Numerical Solution

In order to solve the differential equations (7) and (8), along with boundary condition given by equation (9), we assume that

$$F = F_0(\eta) + \xi F_1(\eta) + \xi^2 F_2(\eta) + \ldots \qquad \ldots(13)$$

$$G = G_0(\eta) + \xi G_1(\eta) + \xi^2 G_2(\eta) + \ldots \qquad \ldots(14)$$

Upon substituting expressions (13) - (14) into the differential equations (7)-(9) and equating the coefficients of like powers of ξ, we obtain a set of ordinary differential equations. These are not reproduced here in the interest of conserving space.

The governing equations for the velocity and temperature functions have been solved on the computer using the fourth-order, Runge-Kutta numerical integration procedure in conjunction with shooting techniques.

The gravity $g(\xi)$ is assumed to follow the form

$$g(\xi) = 1 \pm \xi, \; 0 \leqslant \xi \leqslant 1 \qquad \ldots(15)$$

where the positive sign is taken for the cold plate and negative sign for the hot plate, respectively.

4. Results and Discussion

4.1 *Isothermal Porous Plate*

In this case S_w = constant. The series solutions by Lienhard *et al.*[9] for the parameters C_f and Nu can be written in the form

$$C_f = 4\left(\frac{Gr_x}{4}\right)^{3/4}\left[\sum_{n=0}^{\infty}(-1)^{nl}\,\frac{F''_n(0)\xi^n}{n!}\right] \qquad \ldots(16)$$

$$Nu = \left(\frac{Gr_x}{4}\right)^{1/4}\left[-\sum_{n=0}^{\infty}(-1)^{nl}\,\frac{\theta_n(0)}{n!}\,\xi_n\right] \qquad \ldots(17)$$

where $l = 0$ for positive sign in equation (15) as $l = 1$ for minus sign in the same equation. For comparison purpose, the calculates skin-friction and Nusselt number for cold and hot plates are given in Tables I and II respectively at some porosity parameter F_w.

Figure 1 presents the dimensionless velocity F' and the dimensionless temperature G for the isothermal plate (S_w = const.) at different values of F_w. It is quite clear from this figure that an increase in porosity parameter F_w leads to a decrease in F'. However, near the plate ($\eta \leqslant 0.4$) the temperature shows a decrease as F_w increases.

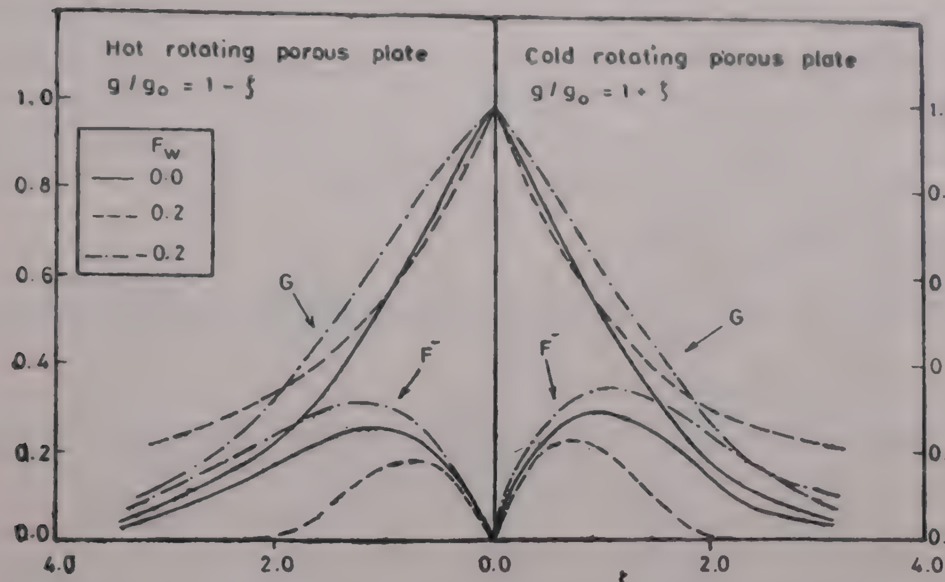

FIG. 1. Temperature and velocity functions for the free convection along rotating isothermal porous plates with $Pr = 0.7$ and $\xi = 0.2$.

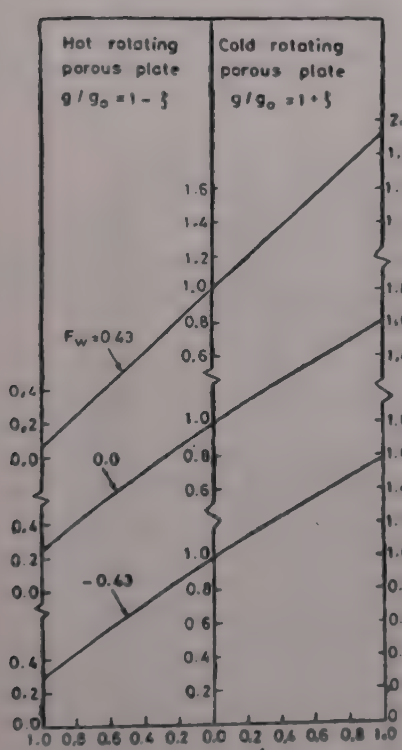

FIG. 2. Comparison of $F''(\xi, 0)/F''(0,0)$ for the free convention along rotating isothermal porous plates, $Pr = 0.7$.

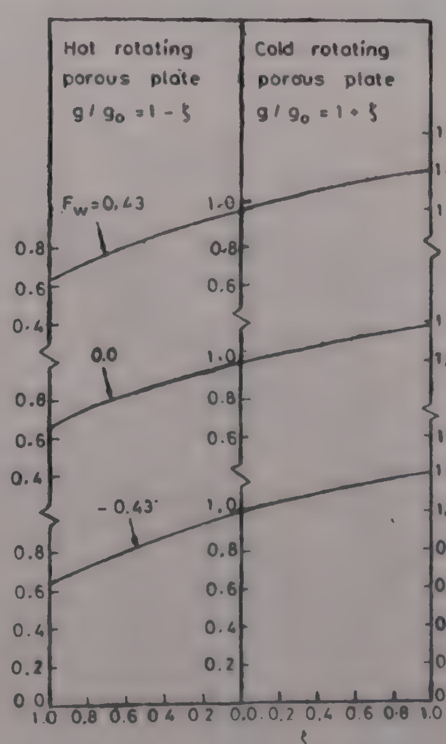

FIG. 3. Comparison of $G'(\xi, 0)/G'(0,0)$ for the free convention along rotating isothermal porous plates, $Pr=0.7$.

Figures 2 and 3 refer to the relative changes of the local skin-friction $F''(\xi, 0)/F''(0,0)$ and the local heat transfer $G'(\xi, 0)/G'(0,0)$ for representative values of ξ and

different values of F_w. For the cold plate the local skin-friction and the local heat transfer increase with the increase of ξ, but for the hot plate the effect of ξ is just the opposite. Also, the increase in the porosity parameter F_w leads to an increase in the local skin friction. The local heat transfer are identical. As is expected, $g(\xi)$ has a significant effect on C_f and Nu as depicted by Figs. 2 and 3.

4.2. *Nonisothermal Porous Plate*

In this case, the class of wall temperature distributions defined, for instance as

$$S_w(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + \ldots \qquad \ldots(18)$$

We shall assign values to the constants a_i. Thus the problem is defined for the following cases :

$$\text{I } S_w = 1 - \tfrac{1}{2}\xi$$

$$\text{II } S_w = 1 + \tfrac{1}{2}\xi$$

$$\text{III } S_w = 1 - \tfrac{1}{2}\xi + \tfrac{1}{4}\xi^2. \qquad \ldots(19)$$

Figures 4 and 5 present the results for the skin-friction and heat transfer for the cases I–III evaluated at various ξ locations for different values of porosity parameter. Also, the profiles in the particular case of isothermal plates ($S_w =$ const.) are also

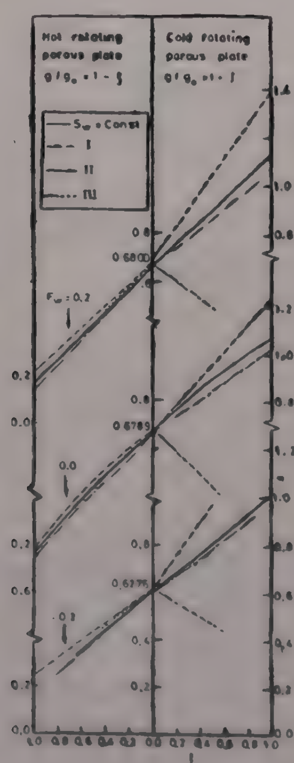

FIG. 4. Skin-fraction results for nonisothermal rotating porous plates.

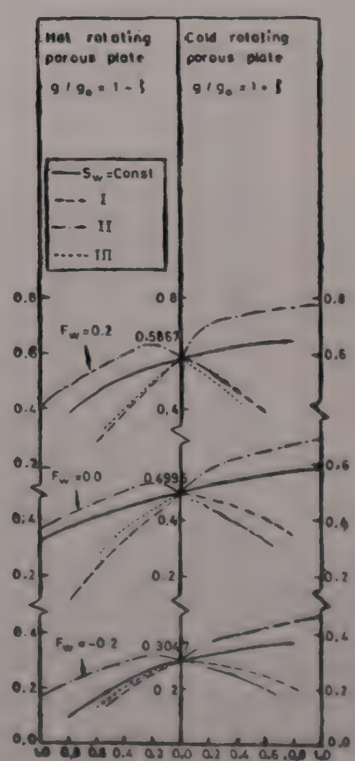

FIG. 5. Heat transfer results for nonisothermal rotating porous plates, $Pr = 0.7$.

TABLE I

Summary of numerical results for an isothermal cold rotating porous plate when $P_r = 0.7$

ξ	$C_f/4\ (Gr_x/4)^{3/4}$			$Nu/\ (Gr_x/4)^{1/4}$		
	$F_w \equiv -0.43$	0,0	0.43	—0.43	0.0	0.43
0.0	0.6630	0.6789	0.5579	0.2898	0.4995	0.8812
0.0375	0.6788	0.6958	0.5772	0.2929	0.5043	0.8875
0.1034	0.7064	0.7250	0.6111	0.2980	0.5124	0.8976
0.1488	0.7251	0.7449	0.6344	0.3015	0.5178	0.9036
0.2426	0.7632	0.7853	0.6826	0.3084	0.5283	0.9140
0.3539	0.8074	0.8319	0.7397	0.3160	0.5398	0.9224
0.4825	0.8571	0.8841	0.8056	0.3242	0.5518	0.9269
0.5713	0.8905	0.9191	0.8510	0.3294	0.5593	0.9268
0.6003	0.9013	0.9304	0.8659	0.3310	0.5616	0.9261
0.6565	0.9219	0.9519	0.8946	0.3310	0.5616	0.9261
0.7710	0.9631	0.9948	0.9530	0.3398	0.5737	0.8971
0.9451	1.0234	1.0572	1.0417	0.3475	0.5834	0.8971

TABLE II

Summary of numerical results for an isothermal hot rotating porous plate when $P_r = 0.7$

ξ	$C_f/4\ (Gr_x/4)^{3/4}$			$Nu/(Gr_x/4)^{1/4}$		
	$F_w \equiv -0.43$	0.0	0.43	—0.43	0.0	0.43
0.0	0.6630	0.6789	0.5579	0.2898	0.4995	0.8812
0.0375	0.6471	0.6619	0.5385	0.2867	0.9946	0.8743
0.1034	0.6187	0.6316	0.5046	0.2811	0.4857	0.8612
0.1488	0.5989	0.6105	0.4811	0.2772	0.4794	0.8513
0.2426	0.5575	0.5662	0.4327	0.2687	0.4657	0.8286
0.3539	0.5073	0.5123	0.3752	0.2581	0.4485	0.7978
0.4825	0.4479	0.4484	0.3085	0.2452	0.4119	0.7256
0.5713	0.4060	0.4032	0.2625	0.2359	0.4119	0.7256
0.6003	0.3922	0.3883	0.2474	0.2328	0.4068	0.7148
0.6565	0.3652	0.3591	0.2183	0.2328	0.4068	0.7148
0.7710	0.3092	0.2985	0.1587	0.2136	0.3748	0.6453
0.9451	0.2219	0.2037	0.0680	0.1928	0.3396	0.5644

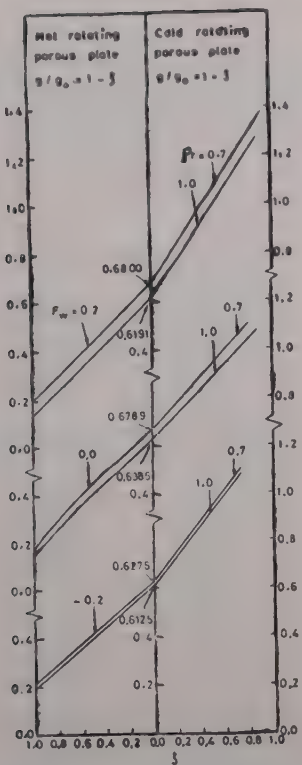

FIG. 6. Skin-friction results for nonisothermal rotating porous plates case I.

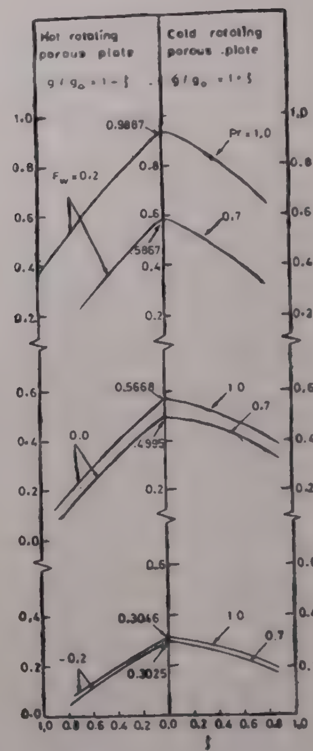

FIG. 7. Heat transfer results for nonisothermal rotating porous plates case I.

plotted at constant ξ distances in Figs. 4 and 5. For all the values of porosity parameter, the skin-friction is greater in case I than in cases II and III of eqns. (19). For the cold plate the skin-friction increases with the increase ξ, but for the hot plate the effect of increasing ξ is to decrease the skin-friction. However for case III the skin-friction decreases with increase of ξ for both the cold and hot plates. For case III also, the increase in F_w leads to an increase in the skin-friction. The results shown in Fig. 5 indicate that the heat transfer for all values of F_w is greater in case II than in cases I and III. Also, an increase in F_w leads to an increase in the heat transfer for all the studied wall temperature.

Figures 6 and 7 show the calculated values of C_f and Nu for case I at successive distances ξ for various porosity parameter with $Pr = 0.7$ and 1.0. The effect of P_r is to decrease C_f and to increase Nu as Pr increase. The effect of Pr becomes predominant at high values of F_w.

REFERENCES

1. R. Eichhorn, *J. Heat Transfer* **82c** (1960), 260–63.
2. E. M. Sparrow, and R. D. Cess, *J. Heat Transfer* **81c** (1960), 287–89.
3. J. H. Merkin, *Int. J. Heat Mass Transfer* **15** (1972), 989.
4. J. F. Clarke, *J. Fluid Mech*, **57** (1973), 45.

5. T. Kao, G. A. Domoto, and H. G. Elrod, *J. Heat Transfer* **99c** (1977), 72–78.
6. J. Aroesty, and J. D. Cole, Rand Corp, Memorandum RM-4620-ARPA (1965).
7. J. H. Merkin, *Int. J. Heat Mass Transfer* **18** (1975) 237.
8. I. Pop and T. Y. Na, *Int. J. Heat Mass Transfer* 26 (1983) 531–38.
9. J. Lienhard, R. Eichhorn, and V. Dihr, *Trans. Am. Soc. Mech. Engrs, Series C, J. Heat Transfer* **94** (1972), 80-86.

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FINITE ELEMENT METHOD IN THE FLOW OF A SECOND-ORDER FLUID BETWEEN TWO POROUS COAXIAL CIRCULAR CYLINDERS

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The present paper deals with the solution of the steady motion of a second-order fluid through the annulus between two coaxial right circular cylinders with suction and injection by finite element method.

Galerkin's method is used to reduce the linear boundary value problem into the corresponding set of algebraic equations. Gauss-Seidel iterative scheme is used to obtain the solution upto the accuracy of the order 1.0×10^{-8}. The flow of the elastico-viscous fluid is taken as a particular case. The effects of second-order forces in the flow and that of suction and injection are discussed in detail and shown graphically.

1. INTRODUCTION

The study of flows in the presence of porous walls has many applications in the field of engineering, technology and biophysics. The viscous flow past a circular cylinder and like problems are studied by many researchers[1-4]. Datta[5] has solved the problem of steady motion of an idealized visco-elastic liquid through an annulus between coaxial circular cylinders and between two parallel plane boundaries, and flow of a non-Newtonian fluid through an annulus with porous walls. Sharma[6] has extended the problems considered in [4,5] for a second-order fluid.

In the present paper we have investigated the steady motion of an incompressible second-order fluid when it flows through the annulus between two coaxial right circular cylinders with suction and injection[6] by using finite element method. It is assumed that the rate of fluid injection at one of the surface is the same as that of withdrawal at the other, and the inner surface is moving with a constant velocity parallel to itself. The effects of the second-order forces in the flow and that of suction and injection represented by dimensionless parameters T_1, T_2 and R respectively have been discussed and shown graphically.

2. FORMULATION OF THE PROBLEM

Coleman and Noll[7] have suggested following constitutive equation for an incompressible second-order fluid:

$$\tau_{ij} = -p_{ij} + 2\mu_1 d_{ij} + 2\mu_2 e_{ij} + 4\mu_3 d^{\alpha}_{i} d_{\alpha j} \qquad \ldots(2.1)$$

where

$$d_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i})$$

and

$$e_{ij} = \tfrac{1}{2}(a_{i,j} + a_{j,i} + 2 v^{m}_{,i} v_{m,j}) \qquad \ldots(2.2)$$

p denotes the hydrostatic pressure and τ_{ij} is the stress-tensor; while v_i and a_i denote velocity and acceleration vectors. μ_1, μ_2 and μ_3 are the coefficients of Newtonian-viscosity, elastico-viscosity and cross-viscosity respectively. Equation (2.1) the momentum equation

$$\rho\left(\frac{\partial v^i}{\partial t} + v_{i,j} v^j\right) = \tau^{j}_{i,j} \qquad \ldots(2.3)$$

and the equation of continuity for steady motion

$$v^{i}_{,i} = 0 \qquad \ldots(2.4)$$

where ρ is the fluid-density and a suffix following a comma denotes covariant differentiation, form the set of governing equations.

Markovitz[8] has shown that 5.46 per cent solution of polyisobutylene in cetane at 30°C behaves as a second-order fluid. The values of the material constants μ_1, μ_2 and μ_3 are calculated by him are 18.5, −0.2 and 1.0 respectively (all expressed in C. G. S. system of units). On this basis the ratio of μ_2 and μ_3 has been taken as −0.2 for the purpose of numerical calculations.

We work with cylindrical coordinates (r, θ, z) and choose the axis of boundaries involved to be along the axis of z. The velocity field for the problem can be chosen in the following form :

$$u = u(r),\ v = 0,\ w = w(r). \qquad \ldots(2.5)$$

In order that the velocity field (2.5) is compatible with continuity criterion we must have :

$$\frac{d}{dr}(ur) = 0$$

which gives on integration

$$ur = c, \text{ a constant.} \qquad \ldots(2.6)$$

Using eqns. (2.1) and (2.2) and velocity field (2.5) we have the physical components of stress tensor as follows :

$$\tau_{rr} = -p + 2\mu_1 \frac{du}{dr} + 2\mu_2\left[u\frac{d^2u}{dr^2} + 2\left(\frac{du}{dr}\right)^2 + \left(\frac{dw}{dr}\right)^2\right]$$

$$+ \mu_3\left[4\left(\frac{du}{dr}\right)^2 + \left(\frac{dw}{dr}\right)^2\right] \qquad \ldots(2.7)$$

$$\tau_{\theta\theta} = -p + 2\mu_1\frac{u}{r} + 2\mu_2\left[\frac{u}{r}\frac{du}{dr} + \frac{u^2}{r^2}\right] + 4\mu_3\frac{u^2}{r^2} \qquad \ldots(2.8)$$

$$\tau_{zz} = -p + \mu_3\left(\frac{dw}{dr}\right)^2 \qquad \ldots(2.9)$$

$$\tau_{rz} = \mu_1\frac{dw}{dr} + \mu_2\left[u\frac{d^2w}{dr^2} + \frac{du}{dr}\cdot\frac{dw}{dr}\right] + 2\mu_3\frac{du}{dr}\cdot\frac{dw}{dr} \qquad \ldots(2.10)$$

$$\tau_{r\theta} = \tau_{\theta z} = 0 \qquad \ldots(2.11)$$

and equations of motion for steady case are :

$$\rho u\frac{du}{dr} = \frac{d}{du}\tau_{rr} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) \qquad \ldots(2.12)$$

and

$$\rho u\frac{dw}{dr} = \frac{d}{dr}\tau_{rz} + \frac{1}{r}\tau_{rz}. \qquad \ldots(2.13)$$

Let us suppose that z-axis is along the common-axis of the cylinders having radii a and b ($a > b$). If the inner cylinder moves with a constant velocity along the direction of the axis and the outer cylinder is fixed, the boundary conditions for the problem are :

$$u = U,\ w = 0 \quad \text{at } r = a$$

$$= \bar{U},\ w = W \quad \text{at } r = b. \qquad \ldots(2.14)$$

From (2.6) and (2.14)

$$aU = ur = \bar{U}b = \text{a constant}$$

or,

$$u = \frac{aU}{r} = \frac{b\bar{U}}{r} = \frac{U}{\eta}$$

where $\eta\left(=\frac{r}{a}\right)$ is dimensionless distance.

Equation (2.13) can be written as :

$$\rho aU\frac{dw}{dr} = \frac{d}{dr}[r\,\tau_{rz}]$$

or,

$$\tau_{rz} = \frac{\rho a U}{r}(w + D) \qquad ...(2.15)$$

where D is a constant. Eliminating τ_{rz}, eqns. (2.10) and (2.15) give :

$$T_1 \frac{d^2 w}{d\eta^2} + \frac{1}{\eta}(\eta^2 - T_1 - 2T_2)\frac{dw}{d\eta} - R(w + D) = 0 \qquad ...(2.16)$$

where $T_1 = \mu_2 U/\mu_1 a$ and $T_2 = \mu_3 U/\mu_1 a$ are dimensionless elastico-viscous and cross-viscous parameters; $R = Ua\rho/\mu_1$ is dimensionless suction parameter.

The boundary conditions on w can be written as :

$$w = 0 \quad \text{when } \eta = 1, \qquad ...(2.17)$$

$$w = W \quad \text{when } \eta = \sigma$$

where σ $(= b/a)$ is ratio of the two radii.

3. Solution of the Problem

Let $w^* = \dfrac{w}{W}$.

After dropping the asteriks, eqns. (2.16) and (2.17) can be written as :

$$T_1 \frac{d^2 w}{d\eta^2} + \frac{1}{\eta}\left(\eta^2 - T_1 - 2T_2\right)\frac{dw}{d\eta} - R(w + D) = 0 \qquad ...(3.1)$$

and

$$w = 0 \quad \text{when } \eta = 1, \qquad ...(3.2)$$

$$w = 1 \quad \text{when } \eta = \sigma.$$

Let the solution of eqn. (3.1) be

$$w = \sum_{i=1}^{M} \phi_i(\eta)\, w_i \qquad ...(3.3)$$

where $\phi_i(\eta)$ is linear shape function and defined as

$$\phi_i(\eta) = \begin{cases} \dfrac{\eta_{i+1} - \eta}{h} & \text{when } \eta_i \leqslant \eta \leqslant \eta_{i+1} \\ \dfrac{\eta - \eta_{i-1}}{h} & \text{when } \eta_{i-1} \leqslant \eta \leqslant \eta_i \\ 0 & \text{otherwise,} \end{cases}$$

where h is the step-length along η.

Putting the values of first and second derivatives of w and using eqn. (3.3), we get the error term as

$$E(\eta) = T_1 \sum_{i=1}^{M} \phi_i''(\eta)\, w_i + \sum_{i=1}^{M} \frac{1}{\eta_i}\left(\eta_i^2 - T_1 - 2T_2\right)\phi_i'(\eta)\, w_i - R$$

$$\left(\sum_{i=1}^{M} \phi_i(\eta)\, w_i + D\right). \qquad ...(3.4)$$

Applying Galerkin finite element method we get

$$\int_{\sigma}^{1} E(\eta).\ \phi_k\, d\eta = 0,\ K = 2\ (1)\ N \qquad ...(3.5)$$

$$\left[-\frac{T_1}{h} - \left(\frac{\eta_{K-1}^2 - T_1 - 2T_2}{2\eta_{K-1}}\right) - \frac{Rh}{6}\right] w_{K-1} + \left(\frac{2T_1}{h} - \frac{2}{3}Rh\right) w_K$$

$$+\left[-\frac{T_1}{h} + \left(\frac{\eta_{K-1}^2 - T_1 - 2T_2}{2\eta_{K+1}}\right) - \frac{Rh}{6}\right] w_{K+1} = RDh,$$

$$K = 2\ (1)\ N \qquad ...(3.6)$$

and boundary conditions (3.2) give

$$w(1) = 0 \quad \text{wnen } \eta = 1,$$

$$w(\sigma) = 1 \quad \text{when } \eta = \sigma. \qquad ...(3.7)$$

Equation (3.6) in matrix form can be written as

$$\begin{bmatrix} b & c_1 & & & \\ a_2 & b & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{N-2} & b & c_{N-2} \\ & & & a_{N-1} & b \end{bmatrix} \begin{bmatrix} w_2 \\ w_3 \\ \vdots \\ w_{N-1} \\ w_N \end{bmatrix} = \begin{bmatrix} RDh - a_1 w_1 \\ RDh \\ \vdots \\ RDh \\ RDh - c_{N-1} w_{N+1} \end{bmatrix}$$

...(3.8)

where

$$a_i = -\frac{T_1}{h} - \left(\frac{\eta_{i-1}^2 - T_1 - 2T_2}{2\eta_{i-1}}\right) \frac{Rh}{6}$$

$$b = \frac{2T_1}{h} - \frac{2}{3} Rh$$

$$c_i = -\frac{T_1}{h} + \left(\frac{\eta_{i+1}^2 - T_1 - 2T_2}{2\eta_{i+1}}\right) - \frac{Rh}{6}, \; i = 1\,(1)\,N-1$$

or,

$$W_K^{(S+1)} = \frac{1}{b}\left(RDh - a_{K-1}\, w_{K-1}^{(S)} - c_{K-1}\, w_{K+1}^{(S)}\right) K = 2\,(1)\,N. \quad \ldots(3.9)$$

Computations are made for $W_K^{(S+1)}$ $(K = 2\,(1)\,N)$ by using the convergence criteria to achieve the desired accuracy (1.0×10^{-8})

Particular Cases :

(i) by putting $T_2 = 0$, we get the results for the elastico-viscous fluid.

(ii) by putting the boundary conditions as

$w = 0$ when $\eta = 1$

$w = 0$ when $\eta \to \infty$.

The results are matched with Sharma *et al.*[9].

4. Conditions for Convergence of the Iterative Process

Equation (3.8) can be written as

$$\begin{aligned} w_2 &= F_1\,(w_1, w_3) \\ w_3 &= F_2\,(w_2, w_4) \\ &\vdots \\ w_N &= F_{N-1}\,(w_{N-1}, w_{N+1}). \end{aligned} \quad \ldots(4.1)$$

Then if $w_1^{(0)}, w_2^{(0)}, \ldots, w_{N+1}^{(0)}$ be the approximate values of $w_1, w_2 \ldots, w_{N+1}$ improved values are found by the following steps :

$$\begin{aligned} w_2^{(1)} &= F_1\left(w_1^{(0)}, w_3^{(0)}\right) \\ w_3^{(1)} &= F_2\left(w_2^{(0)}, w_4^{(0)}\right) \\ &\vdots \\ w_N^{(1)} &= F_{N-1}\left(w_{N-1}^{(0)}, w_{N+1}^{(0)}\right). \end{aligned} \quad \ldots(4.2)$$

Subtracting this system from eqn. (4.1), we have

$$w_2 - w_2^{(1)} = F_1\,(w_1, w_3) - F_1\left(w_1^{(0)}, w_3^{(0)}\right)$$

$$w_3 - w_3^{(1)} = F_2(w_2, w_4) - F_2\left(w_2^{(0)}, w_4^{(0)}\right)$$

$$\vdots$$

$$w_N - w_N^{(1)} = F_{N-1}(w_{N-1}, w_{N+1}) - F_{N-1}\left(w_{N-1}^{(0)}, w_{N+1}^{(0)}\right). \qquad \ldots(4.3)$$

Now applying to the right-hand side of the system (4.3) the mean value of theorem we have

$$F_1(w_1, w_3) - F_1\left(w_1^{(0)}, w_3^{(0)}\right) = \left(w_1 - w_1^{(0)}\right)\frac{\partial \bar{F}_1}{\partial w_1} + \left(w_3 - w_3^{(0)}\right)\frac{\partial \bar{F}_1}{\partial w_3}$$

where

$$\frac{\partial \bar{F}_1}{\partial w_1} = \frac{\partial F_1\left[w_1^{(0)} + \theta\left(w_1 - w_1^{(0)}\right), w_3^{(0)} + \theta\left(w_3 - w_3^{(0)}\right)\right]}{\partial w_1}, \quad 0 \leqslant \theta \leqslant 1$$

and

$$\frac{\partial \bar{F}_1}{\partial w_2} = \frac{\partial F_1\left[w_1^{(0)} + \theta\left(w_1 - w_1^{(0)}\right), w_3^{(0)} + \theta\left(w_3 + w_3^{(0)}\right)\right]}{\partial w_3}, \quad 0 \leqslant \theta \leqslant 1.$$

In a similar manner we get

$$F_2(w_2, w_4) - F_2\left(w_2^{(0)}, w_4^{(0)}\right) = \left(w_2 - w_2^{(0)}\right)\frac{\partial \bar{F}_2}{\partial w_2} + \left(w_4 - w_4^{(0)}\right) \times \frac{\partial \bar{F}_2}{\partial w_4}$$

$$F_3(w_3, w_5) - F_3\left(w_3^{(0)}, w_5^{(0)}\right) = \left(w_3 - w_3^{(0)}\right)\frac{\partial \bar{F}_3}{\partial w_3} + \left(w_5 - w_5^{(0)}\right)\frac{\partial \bar{F}_3}{\partial w_5},$$

$$\vdots$$

$$F_{N-1}(w_{N-1}, w_{N+1}) - F_{N-1}\left(w_{N-1}^{(0)}, w_{N+1}^{(0)}\right) = \left(w_{N-1} - w_{N-1}^{(0)}\right)\frac{\partial \bar{F}_{N-1}}{\partial w_{N-1}} + \left(w_{N+1} - w_{N+1}^{(0)}\right) \times \frac{\partial \bar{F}_{N-1}}{\partial w_{N+1}}.$$

Substituting these expressions for the right-hand members of eqn. (4.3), it becomes

$$w_2 - w_2^{(1)} = \left(w_1 - w_1^{(0)}\right)\frac{\partial \bar{F}_1}{\partial w_1} + \left(w_3 - w_3^{(0)}\right)\frac{\partial \bar{F}_1}{\partial w_3}$$

$$w_3 - w_3^{(1)} = \left(w_3 - w_3^{(0)}\right)\frac{\partial \bar{F}_2}{\partial w_3} + \left(w_4 - w_4^{(0)}\right)\frac{\partial \bar{F}_2}{\partial w_4}$$

$$\vdots$$

$$w_N - w_N^{(1)} = \left(w_{N-1} - w_{N-1}^{(0)}\right)\frac{\partial \bar{F}_{N-1}}{\partial w_{N-1}} + \left(w_{N+1} - w_{N+1}^{(0)}\right)\frac{\partial \bar{F}_{N-1}}{\partial w_{N+1}}. \quad ...(4.4)$$

Adding these equations and considering only the absolute values of the several quantities, we have

$$\left|w_2 - w_2^{(1)}\right| + \left|w_3 - w_3^{(1)}\right| + ... + \left|w_N - w_N^{(1)}\right| \leqslant \left|w_1 - w_1^{(0)}\right|\left|\frac{\partial \bar{F}_1}{\partial w_1}\right|$$

$$+ \left|w_2 - w_2^{(0)}\right|\left|\frac{\partial \bar{F}_2}{\partial w_2}\right| + \left|w_3 - w_3^{(0)}\right|\left|\frac{\partial \bar{F}_1}{\partial w_3}\right|$$

$$+ \left|w_4 - w_4^{(0)}\right|\left|\frac{\partial \bar{F}_2}{\partial w_4}\right| + ... \left|w_{N-1} - w_{N-1}^{(0)}\right|\left|\frac{\partial \bar{F}_{N-4}}{\partial w_{N-1}}\right|$$

$$+ \left|w_{N+1} - w_{N+1}^{(0)}\right|\left|\frac{\partial \bar{F}_{N-1}}{\partial w_{N+1}}\right|. \quad ...(4,5)$$

Finally, as in Scarborough[10], for convergence we have :

$$\left|\frac{\partial F_1}{\partial w_1}\right| + \left|\frac{\partial F_1}{\partial w_3}\right| < 1$$

$$\left|\frac{\partial F_2}{\partial w_2}\right| + \left|\frac{\partial F_2}{\partial w_4}\right| < 1$$

$$\vdots$$

$$\left|\frac{\partial F_{N-1}}{\partial w_{N-1}}\right| + \left|\frac{\partial F_{N-1}}{\partial w_{N+1}}\right| < 1. \quad ...(4.6)$$

Using eqns. (3.8) and (4.1), eqn (4.6) becomes :

$$| b | < | c_1 |$$

$$| b | < | a_k | + | c_k |, k = 2\,(1)\,N - 2$$

$$| b | < | a_{N-1} |. \quad ...(4.7)$$

This is a condition for convergence of the solution.

5. Numerical Values and Discussion

The numerical computations have been made for $T_1 = -0.01, -0.05, -0.10$; $R = -0.4, -0.8, -1.2, 0.10, 0.15\ 0.20$ and $D = -0.5, -1.0, -1.5, 0.05,$

0.10, 0.15. Figure 1 depicts the behoviour of non dimensional component of velocity when T_1 and R is fixed and the constant D is varying. It is noted that the behaviour of w is unaffected with D. The variation of w with suction parameter R when Γ_1 is taken as fixed, is illustrated through Figs. 2 and 3 in case of suction and injection on the wall of outer cylinder. In first case, it is observed that the velocity component decreases as the suction parameter increases on the wall of the outer cylinder. In the case of injection, it decreases near the inner cylinder and increases near the outer cylinder with an increase in R. Figure 4 shows the behaviour of w for different values of cross-viscous parameter T_2 when suction parameter is constant. It is found that the velocity component decreases with an increase in T_1 for suction as well as injection.

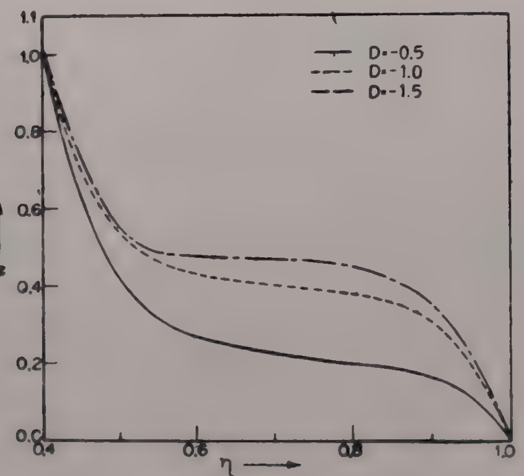

FIG. 1. Variation of Axial Velocity with D for fixed T_1 (= − 0.05) and R (= − 1.2).

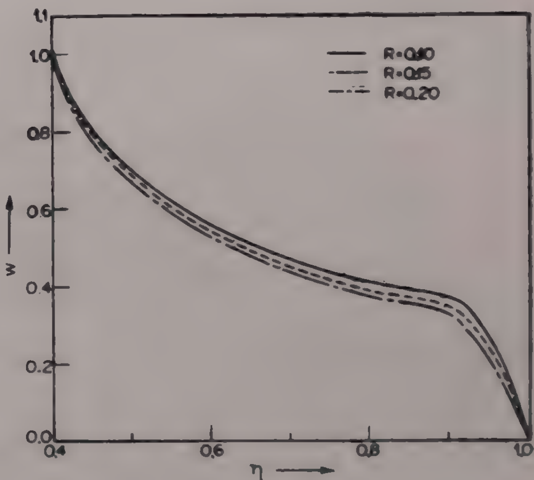

FIG. 2. Variation of Axial Velocity with R (For suction case) for fixed T_1 (= −0.01) and D (= 0.05).

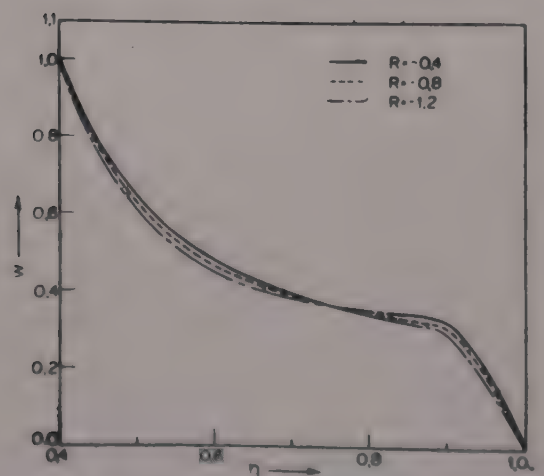

FIG. 3. Variation of Axial Velocity with R (For injection case) for fixed T_1(= − 0.01)and D (= − 0.5).

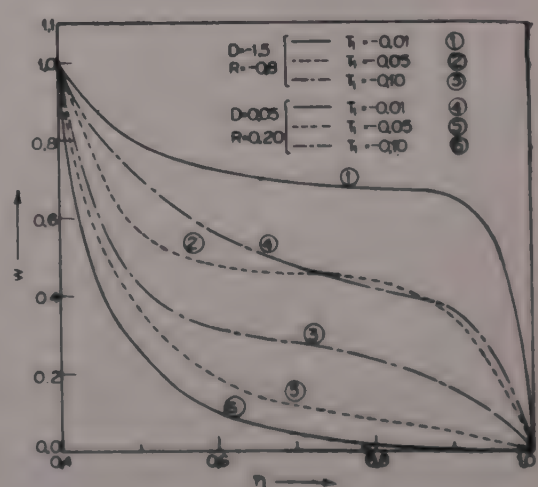

FIG. 4. Variation of Axial Velocity with T_1 (For suction & injection cases for fixed R & D.

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References

1. F. W. Meredith, and A. A. Griffith, *A. R. C.* (1955) 3501.
2. J. M. Key, *A.R. C.* (1948), 2628
3. H. G. Lew, *J. Aero. Sci.* 23 (1958), 276.
4. S. P. Misra, *Ganita*, **14** (1963), 99.
5. S. K. Datta, *J. Phys. Soc. Japan* **16** (1966), 794.
6. H. G. Sharma, *J. Phys. Soc. Japan* **21** (1963), 173.
7. B. D. Coleman, and W. Noll *Arch. Ration Mech. Anal.* **6** (1960) 355.
8. H. Markovitz, Refer to A. C. Srivastava, *J. Fluid Mech.* **17** (1963), 171.
9. H. G. Sharma, Manoj Kumar, and Uma Arora, Numerical solution of steady motion of a second-order fluid past a porous circular cylinder under communication. *Ind. Jour Tech.*
10. J. B. Scarborough, *Numerical Mathematical Analysis* Oxford and IBH Publishing Company, 1966, 301.

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IMPLICIT NUMERICAL SOLUTION OF UNSTEADY EULER EQUATIONS FOR TRANSONIC FLOWS

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In the present paper, an implicit method described which makes use of the differential form of the unsteady Euler equations transformed to a rectangular computational domain. This technique entails a substantial reduction in computation time while, at the same time, retaining second-order accuracy. The method is applied to the case of transonic flow past a 6 per cent symmetric circular arc airfoil and a computer program developed. The results from the software for surface pressure distribution on the airfoil have exhibited excellent agreement with the reported experimental results due to Knechtel for a 0.908 Mach flow, as also computed pressure values through an explicit scheme.

NOMENCLATURE

A, B = Jacobian of F, G

c = Speed of sound

D = Diagonal matrix

e = Total energy per unit volume

F, G = Vector flux (see eqn. (1))

I = Identity matrix

J = Jacobian of transformation

n = Superscript representing advance of time

p = Pressure

R = Used to define equation of body surface

R' = See eqn. (4)

S = Diagonalisation matrix

T = Temperature

t = Time

U = Dependent variable $(\rho, \rho u, \rho v, e)^T$

u, v = Velocity components in Cartesian frame

x, y = Cartesian coordinates

Λ = Eigenvalue matrix

ρ = Density

ξ, η = Transformed spatial coordinates

$\Delta t, \Delta x$ = Elemental increase in $t, x, \ldots$

Subscript

i, j = i, j mesh points in ξ, η direction

$+$ = Forward difference

$-$ = Backward difference

Superscript

$-$ = Predicted value

$'$ = Transformed coordinate.

1. Introduction

Unsteady Euler equations have the advantage of being applicable in cases of supersonic, subsonic and mixed flows. Even in cases where other simplified flow models, such as steady or potential equations, are valid and could be solved without much difficulty, the simulation of the actual flow from the initial conditions is of relevance not only because it is of considerable theorectical interest, but from the point of view of the insight it provides into the flow development. The solutions of these equations can be obtained by using the explicit time-marching method[2,4,6,7]. Madhavan and Swaminathan[2,6,7]. A major disadvantage of such methods, which are quite accurate, is, however, the stringent restriction on the time-step size and the consequent long computation time necessary to obtain convergence.

MacCormack's[4] implicit predictor-correct numerical scheme, which is second-order accurate, is easily applicable and involves a substantial reduction of computation time by relaxing the restriction on the increment in time, may be made use of to overcome the above difficulty. The method involves only the inversion of bidiagonal matrices instead of tridiagonal ones as is normally required in implicit schemes. This technique is extended in this paper to the case of unsteady Euler equations transformed into a rectangular computational domain from the physical domain of any two-dimensional geometry.

The validation of the algorithm and the associated computer program was carried out by comparing the corresponding results obtained through the software with Knechtel's experimental data for surface pressure distribution for transonic flow past a 6 per cent symmetric circular arc airfoil[1]. The computation time reduces to a small fraction (approximately 30%) of that necessary in cases where the explicit scheme is applied to the integral form of the unsteady Euler equations.

2. Governing Equations and Transformation

The two-dimensional unsteady Euler equations in conservation form are given by

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \qquad \ldots(1)$$

where

$$U = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{Bmatrix}$$

$$F = \begin{Bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (e + p)u \end{Bmatrix}$$

$$G = \begin{Bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (e+p)\, v \end{Bmatrix}. \qquad \ldots(2)$$

The numerical integration of these equations in the physical space is difficult owing to the fact that the grid will not be rectangular and consequently a transformation is made use of. It $y = R(x)$ represents the equation of the body surface (eg. airfoil), the transformation $\xi = x$ and $\eta = y - R(x)$ transforms the computational domain into a rectangular one, while eqn. (1) assumes the form

$$\frac{\partial U'}{\partial t} + \frac{\partial F'}{\partial \xi} + \frac{\partial G'}{\partial \eta} = 0. \qquad \ldots(3)$$

Considering the general transformation $\xi = x$ and $\eta = \eta(x, y)$; we have

$$U' = U/J, \; F' = F/J$$

$$G' = \left(\frac{\partial \eta}{\partial x} F + \frac{\partial \eta}{\partial y} G \right) \Big/ J, \; J = \frac{\partial (\xi, \eta)}{\partial (x, y)}.$$

In the present case,

$$U' = U, \quad F' = F$$

$$G' = G - R'(x)\, F. \qquad \ldots(4)$$

For a 6 per cent symmetric circular arc airfoil of chord length unity, the equation of the body surface, referred to rectangular coordinates with the x-axis along the chord length and origin at one end of the chord, is given by

$$x^2 + y^2 - x + 8.3033\, y = 0$$

and consequently the transformed coordinate $\eta = \eta\,(x, y)$ is obtained as

$$\eta = y - R\,(x)$$

$$= y - \{-8.3033 + \sqrt{8.3033^2 - 4x\,(x-1)}\}/2.$$

3. The Implicit Numerical Scheme

Differentiating eqn. (3) partially w. r. t. τ, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial t}\right) = -\frac{\partial}{\partial \xi}\left(A\,\frac{\partial U}{\partial t}\right) - \frac{\partial}{\partial \eta}\left(B\,\frac{\partial U}{\partial t}\right) \qquad \ldots(5)$$

where A and B are, respectively, the Jacobians of F and G' w.r.t. U. Implicitly approximating eqn. (5), we have

$$\frac{\dfrac{\partial U^{n+1}}{\partial t} - \dfrac{\partial U^{n}}{\partial t}}{\Delta t} = -\frac{\partial}{\partial \xi}\left(A\,\frac{\partial U}{\partial t}\right)^{n+1} - \frac{\partial}{\partial \eta}\left(B\,\frac{\partial U}{\partial t}\right)^{n+1}$$

or

$$\left(I + \Delta t\,\frac{\partial}{\partial \xi} A. + \Delta t\,\frac{\partial}{\partial \eta} B.\right)\frac{\partial U^{n+1}}{\partial t} = \frac{\partial U^{n}}{\partial t}. \qquad \ldots(6)$$

Defining

$$\delta U^{n+1} = \frac{\partial U^{n+1}}{\partial t}\,\Delta t$$

and

$$\Delta U^{n} = \frac{\partial U^{n}}{\partial t}\,\Delta t$$

$$\left(I + \Delta t\,\frac{\partial}{\partial \xi} A. + \Delta t\,\frac{\partial}{\partial \eta} B.\right)\delta U^{n+1}_{ij} = \Delta U^{n}_{ij}\,. \qquad \ldots(7)$$

Now an implicit predictor-corrector scheme may be defined for the numerical integration of the transformed eqns. (3) as follows :

Predictor :

$$\Delta U^{n}_{ij} = -\Delta t\left(\frac{D_+}{\Delta \xi}\,F'^{n}_{ij} + \frac{D_+}{\Delta \eta}\,G'^{n}_{ij}\right)$$

$$\left(I - \Delta t \ \frac{D_+}{\Delta\xi} \ |A| .\right)\left(I - \Delta t \ \frac{D_-}{\Delta\eta} \ |B| . \right)\delta \overline{U_{ij}^{n+1}} = \Delta U_{ij}^n$$

$$\overline{U_{ij}^{n+1}} = U_{ij}^n + \delta\, \overline{U_{ij}^{n+1}}.$$

Corrector :

$$\Delta\overline{U_{ij}^{n+1}} = -\Delta t\left(\frac{D_-}{\Delta\xi} \overline{F_{ij}^{\prime n+1}} + \frac{D_-}{\Delta\eta} \overline{G_{ij}^{\prime n+1}}\right)$$

$$\left(I + \Delta t \ \frac{D_-}{\Delta\xi} \ |A| .\right)\left(I + \Delta t \ \frac{D_-}{\Delta\eta} \ |B| .\right) \delta U_{ij}^{n+1} = \Delta\overline{U_{ij}^{n+1}}$$

$$U_{ij}^{n+1} = \frac{1}{2}\left[U_{ij}^n + \overline{U_{ij}^{n+1}} + \delta U_{ij}^{n+1}\right]. \qquad \ldots(8)$$

The matrices $|A|$ and $|B|$ have positive eigenvalues and are related to Jacobians A and B. Let S_ξ, S_η and their inverses denote the matrices that diagonalise A and B. Then, with perfect gas relations, A and B may be arranged as

$$A = S_\xi^{-1} \wedge_A S_\xi \qquad \ldots(9)$$

and

$$B = S_\eta^{-1} \wedge_B S_\eta$$

where

$$S_\xi = \begin{bmatrix} 1 - \dfrac{\alpha\beta}{C^2} & \dfrac{u\beta}{C^2} & \dfrac{v\beta}{C^2} & -\dfrac{\beta}{C^2} \\ -uc + \alpha\beta & C - u\beta & -v\beta & \beta \\ -\dfrac{v}{\rho} & 0 & \dfrac{1}{\rho} & 0 \\ uc + \alpha\beta & -c - u\beta & -v\beta & \beta \end{bmatrix}$$

$$S_\eta = \begin{bmatrix} 1 - \dfrac{\alpha\beta}{C^2} & \dfrac{u\beta}{C^2} & \dfrac{v\beta}{C^2} & -\dfrac{\beta}{C^2} \\ \dfrac{\eta_x v - \eta_y u}{\rho\alpha_2} & \dfrac{\eta_y}{\rho\alpha_2} & -\dfrac{\eta_x}{\rho\alpha_2} & 0 \\ \beta_1\left(\alpha\beta - c\dfrac{\alpha_1}{\alpha_2}\right) & \beta_1\left(\dfrac{\eta_x}{\alpha_2}c - u\beta\right) & \beta_1\left(\dfrac{\eta_y}{\alpha_2}c - v\beta\right) & \beta_1\beta \\ \beta_1\left(\alpha\beta + c\dfrac{\alpha_1}{\alpha_2}\right) & -\beta_1\left(\dfrac{\eta_x}{\alpha_2}c + u\beta\right) & -\beta_1\left(\dfrac{\eta_y}{\alpha_2}c + v\beta\right) & \beta_1\beta \end{bmatrix}$$

$$\Lambda_A = \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u+c & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u-c \end{bmatrix}$$

$$\Lambda_B = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_1 + c\alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_1 - c\alpha_2 \end{bmatrix}$$

$$\alpha_1 = \eta_x u + \eta_y v, \; \alpha_2 = \left(\eta_x^2 + \eta_y^2\right)^{1/2}$$

$$\alpha = \tfrac{1}{2}(u^2 + v^2), \; \beta = \gamma - 1, \; \beta_1 = 1/\sqrt{2}\rho c. \qquad \text{...(10)}$$

The matrices $|A|$ and $|B|$ are now defined by

$$|A| = S_\xi^{-1} D_A S_\xi$$

and

$$|B| = S_\eta^{-1} D_B S_\eta \qquad \text{...(11)}$$

where

$$D_A = \begin{bmatrix} \lambda_{A_1} & 0 & 0 & 0 \\ 0 & \lambda_{A_2} & 0 & 0 \\ 0 & 0 & \lambda_{A_3} & 0 \\ 0 & 0 & 0 & \lambda_{A_4} \end{bmatrix}$$

$$D_B = \begin{bmatrix} \lambda_{B_1} & 0 & 0 & 0 \\ 0 & \lambda_{B_2} & 0 & 0 \\ 0 & 0 & \lambda_{B_3} & 0 \\ 0 & 0 & 0 & \lambda_{B_4} \end{bmatrix} \qquad \text{...(12)}$$

$$\lambda_{A_1} = \lambda_{A_3} = \max\left\{ |u| - \frac{1}{2}\frac{\Delta\xi}{\Delta t}, 0.0 \right\}$$

$$\lambda_{A_2} = \max\left\{ |u+c| - \frac{1}{2}\frac{\Delta\xi}{\Delta t}, 0.0 \right\}$$

$$\lambda_{A_4} = \max\left\{ \mid u - c \mid - \frac{1}{2}\frac{\Delta\xi}{\Delta t}, 0.0 \right\}$$

$$\lambda_{B_1} = \lambda_{B_2} = \max\left\{ \mid \alpha_1 \mid - \frac{1}{2}\frac{\Delta\eta}{\Delta t}, 0.0 \right\}$$

$$\lambda_{B_3} \max\left\{ \mid \alpha_1 + c\alpha_2 \mid - \frac{1}{2}\frac{\Delta\eta}{\Delta t}, 0.0 \right\}$$

$$\lambda_{B_4} = \max\left\{ \mid \alpha_1 - c\alpha_2 \mid - \frac{1}{2}\frac{\Delta\eta}{\Delta t}, 0.0 \right\}. \quad ...(13)$$

For regions of the flow in which Δt satisfies the explicit stability criterion, all λ_A and λ_B vanish and the set of difference equations (8) just reduce to corresponding explicit method.

4. Comruter Program and Results

A computer program was developed in FORTRAN IV language for carrying out the above computations and was made operational on the CDC CYBER170/730 Computer of VSSC. The experimental results used for comparison were those due to Knechtel for Mach 0.908 transonic flow over 6 per cent symmetric circular arc airfoil[1]. The upstream, downstream and top mesh boundary conditions were held fixed at uniform freestream flow in the expectation that the flow near the airfoil will converge before the disturbances reach those points which are held at 10 chord lengths away. The surface pressure on the airfoil was taken to be equal to that at the mesh point immediately above it.

A 48 $\times$ 32 mesh system was used in the program. Along the x-direction, a two-mesh system of a stretched mesh (of 10 points) fore and aft of the airfoil and a uniform mesh of 28′ points on the airfoil was made use of. Along the y-direction, a stretched mesh was applied.

The data processing rate was about 8 secs. for iteration, i.e. 2.5 $\times$ 10^{-3} secs. per grid point including prediction and correction, that is, on an average, about 60 per cent more than that for the explicit scheme. Thus, when a CFL number of 5 was used, the implicit scheme took around one-third of the time for the explicit method. CFL numbers up to about 30 could be used without loss of accuracy. A maximum number of 64 iterations were required to obtain convergence when the difference in consecutive flow parameter values becomes less than about 0.1 per cent (Shang and Hankey[5]) and this took around 4.0 minutes on the computer. The corresponding time for the explicit scheme applied to an integral from of the unsteady Euler equations for the same problem was approximately 11.0 min.

Figure 3 presents the computed surface pressure values on the airfoil for Mach 0.908 flow together with the corresponding experimental values due to Knechtel and,

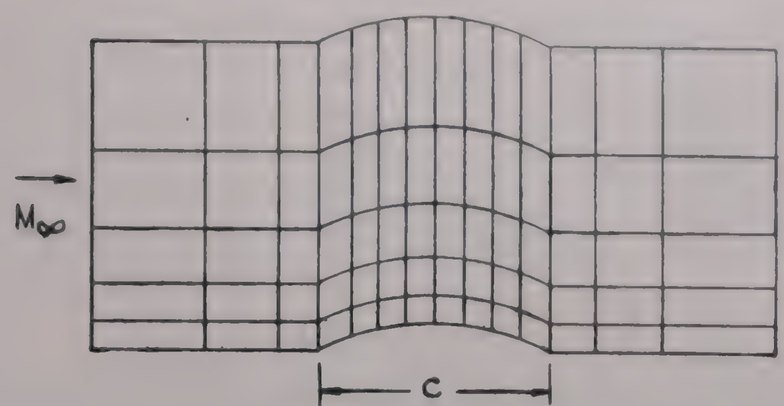

FIG. 1. Mesh about typical circular *ARC* airfoil.

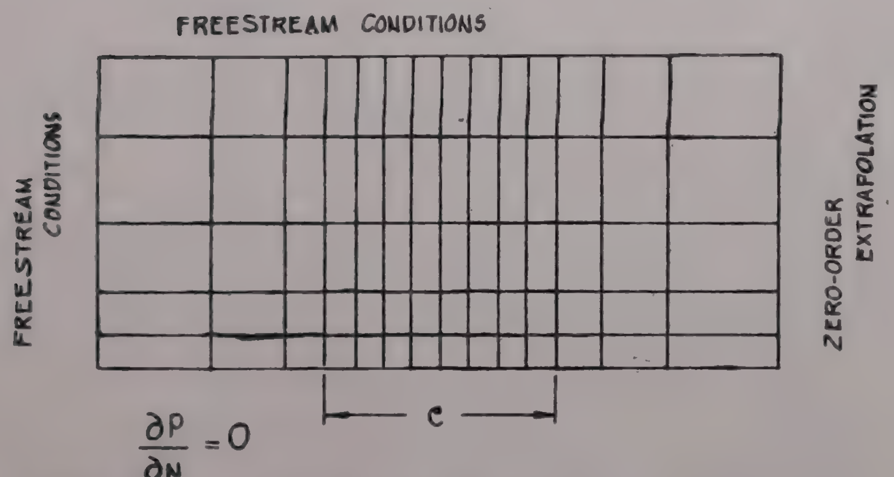

FIG. 2. Mesh system and boundary conditions in transformed computational plane.

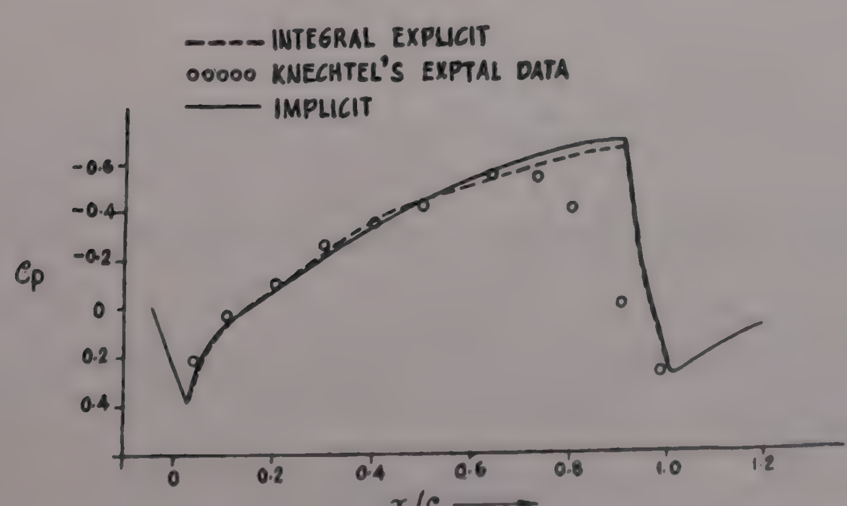

FIG. 3. Comparison of pressure distribution for circular *ARC* airfoil ($M_\infty = 0.908$)

as can be readily observed, the comparison is quite good. The dotted line indicates surface pressure values computed by the corresponding cxplicit scheme. Figure 4 and

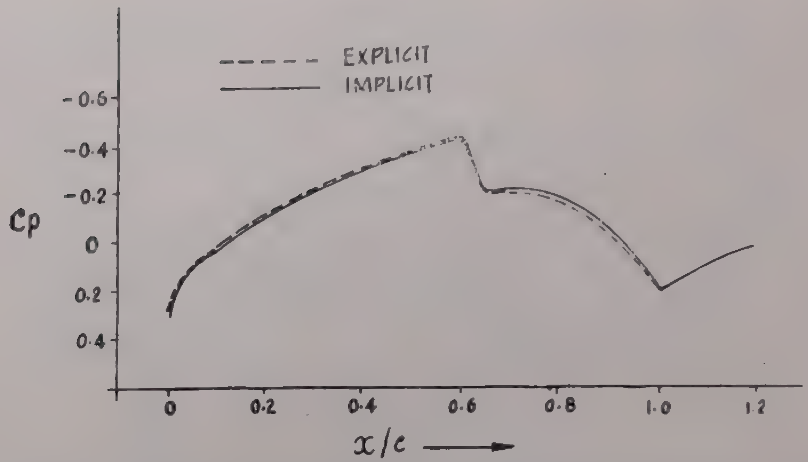

FIG. 4. Pressure distribution for circular *ARC* airfoil (M_∞ = 0.8).

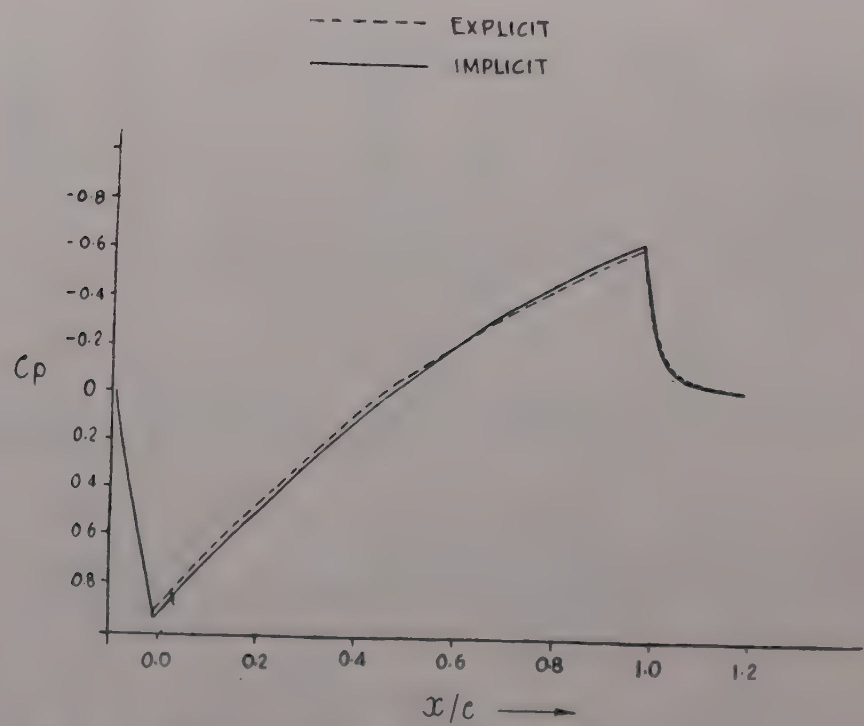

FIG. 5. Pressure Distribution for circular *ARC* airfoil (M_∞ = 1.2).

5, respectively, exhibit the computed surface pressure for Mach 0.8 and Mach 1.2 flows.

5. Conclusion

An implicit numerical method for solving unsteady Euler equations transformed to a rectangular computational domain is presented in this paper. The method was applied for studying transonic flow past a 6 per cent symmetric circular are airfoil. and a computer program developed for the purpose. The method entails a substantial

reduction in computation time as compared to the corresponding explicit scheme, while retaining second-order accuracy. A comparison of the results for surface pressure distribution obtained through the software for a 0.908 Mach transonic flow past circular arc airfoil with reported experimental values was quite good. Work is presently under way on solving Navier-Stokes equations using an analogous technique for the transonic shock wave boundary layer interaction problem.

REFERENCES

1. E. D. Knechtel, Experimental investigation at transonic speeds of pressure disrribution over wedge and circular arc airfoil sections and evaluation of perforated wall interference-*NASA TND*-15, 1959.
2. N. S. Madhavan, and V. Swaminathan, *Proc. 11th National Conf. Fluid Mech. Fluid Power BHEL, Hyderabad.* CFI-CF5, 1982.
3. R. W. MacCormack, *Numerical Methods in Fluid Dynamics* (ed.: M. Holt). Springer-Verlag, 8 (1971), 151-63.
4. R. W. MacCormack, *AIAA Jl* 20 (1982), 1275–81
5. J. S. Shang, and W. L. Hankey, *AIAA 13th Aerospace Sciences Meeting*, *AIAA Paper* 75-3, 4, 1975.
6. V. Swaminathan, and N. S. Madhavan, *Proc. 3rd Int. Conf. Num. Methods for Laminar and Turbulent Flow*, University of Washington, Aug. 8-11, 1983.
7. V. Swaminathan, and N. S. Madhavan, In: *Aspects of Mechanics, Lecture Notes in Applied Mathematics*, (ed.: D. K. Sinha) South Asian Pub. Co. Delhi; 1985, pp. 53–59.

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